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A Semiclassical Parabolic System Related to F_4

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This paper is part of a greater work which consists of the classification of a certain class of semiclassical parabolic systems of rank 4 (see [W3]).

DEFINITION 1. Let G be any group. A *semiclassical parabolic system* for G is a set $\{P_i | i \in I\}$, $I = \{1, \dots, n\}$, of subgroups of G with the following properties:

(i) $G = \langle P_1, \dots, P_n \rangle$ and $G \neq G_i = \langle P_j | j \in I \setminus \{i\} \rangle$ for all $i \in I$.

(ii) There exists a finite subgroup $S \leq B = \bigcap_{i=1}^{n} P_i$ such that $S \in Syl_2(P_i) \cap Syl_2(P_{ij})$ for all $i, j \in I$, where $P_{ij} = \langle P_i, P_j \rangle$.

(iii) For all $i \in I$, P_i/B_{P_i} is a rank-1 Lie group defined over a field of char 2 with Borel subgroup B/B_{P_i}.

(iv) For all $i, j \in I$, $i \neq j$, either $P_{ij} = P_i P_j$ or $P_{ij}/B_{P_{ij}}$ is a rank-2 Lie group defined over a field of char 2 or $P_{ij}/B_{P_{ij}} \cong 3A_6$ or $3\Sigma_6$ and the last case occurs at least once (otherwise the system is called *classical*).

(v) $B_G = \bigcap_{g \in G} B^g = 1$.

We call the subgroups P_i the *minimal parabolics*, the G_i the *maximal parabolics*, and n the *rank* of the parabolic system.

To each parabolic system we associate a diagram over I in the following way:

- if $P_{ij} = P_i P_j$ then i, j are not connected,
- if $P_{ij}/B_{P_{ij}}$ is a rank-2 Lie group, we take the corresponding Dynkin diagram for the edge between i and j,
- if $P_{ij}/B_{P_{ij}} L \cong 3A_6$ or $3\Sigma_6$, we take the symbol $\overset{i\ \sim\ j}{\circ\!\!=\!\!=\!\!\circ}$.

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Parabolic systems are closely related to flag-transitive geometries and can be viewed as a certain generalization of buildings (for details see, e.g., [Bue] or [Pas]). The class of semiclassical parabolic systems seems to be interesting because a couple of the sporadic simple groups possess such systems (see, e.g., [RS]).

Here we show the uniqueness of the amalgam of maximal parabolics of a certain semiclassical parabolic system of rank 4 which we call a system *of type* $\tilde{F}_4$. This system belongs to the diagram $\overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{2}{\circ}\!\overset{\sim}{=\!=}\!\overset{3}{\circ}\!\!-\!\!\!-\!\!\overset{4}{\circ}$ and we assume that $G_1/B_{G_1} \cong G_4/B_{G_4} \cong 3^7Sp_6(2)$. The reason why we say that such a system is of type $\tilde{F}_4$ is that it will turn out that it is closely related to the finite simple group of Lie type $F_4(2)$.

THEOREM. *Let G be a group possessing a semiclassical parabolic system of type $\tilde{F}_4$. Then there exists a homomorphism φ of G onto the finite simple group of Lie type $F_4(2)$ which maps $\{G_1, G_2, G_3, G_4\}$ onto the maximal parabolic subgroups of $F_4(2)$.*

Remark 1. (i) It was shown in [Hei] that the only groups which have a semiclassical parabolic system with diagram $\circ\!\!-\!\!\!-\!\!\circ\!\overset{\sim}{=\!=}\!\circ$ are a nonsplit extension $3^7Sp_6(2)$ and the sporadic simple groups M_{24} and He. In [FS] it was shown that the systems for M_{24} and He cannot be involved in a rank-4 system with the above diagram. Hence the assumption on G_1 and G_4 can be dropped modulo those results.

(ii) It was shown in [W1] that if $\ker \varphi$ is abelian, then it is of size 3^{833} and G is a nonsplit extension $3^{833}F_4(2)$. In [W2] an example of such a nonsplit extension was constructed, which shows that there indeed exists a parabolic system of type $\tilde{F}_4$.

The paper is structured as follows: The needed background results, especially about modules and the amalgam method, are contained in Section 1. Sometimes we will present them in a more general version than needed in this paper in order to be able to apply them to other cases as well. In Section 2 we show that the structure of the parabolics G_1, G_4, and G_{14} globally resembles that of the group $F_4(2)$ and in Section 3 we give explicit generators and relations for G. Those generators and relations will be closely related to the Steinberg presentation of $F_4(2)$ and easily show the existence of the desired homomorphism φ. In fact, if $W = \langle r_1, r_2, r_3, r_4\rangle$ is the Weyl group of $F_4(2)$ corresponding to the diagram $\overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{2}{\circ}\!=\!=\!\overset{3}{\circ}\!\!-\!\!\!-\!\!\overset{4}{\circ}$, we will show that we just have to replace the relation $(r_2r_3)^4 = 1$ by the relation $(r_2r_3)^{12} = 1$ and to add a few more relations to ensure that $\langle P_2, P_3\rangle/O_2(\langle P_2, P_3\rangle) \cong 3\Sigma_6$ instead of Σ_6.

In the following, if not stated explicitly otherwise, G always denotes a group with a semiclassical parabolic system $\{P_1, \dots, P_n\}$, $G_i = \langle P_j | j \neq i\rangle$,

$K_i = B_{G_i}$, $G_{ij} = \langle P_k | k \neq i, j \rangle$, $K_{ij} = B_{G_{ij}}$, N_i is the full preimage of $O_3(G_i/K_i)$ in G_i, N_{ij} is the full preimage of $O_3(G_{ij}/K_{ij})$ in G_{ij}, $Z_i = \langle \Omega_1(Z(S))^{G_i} \rangle$, and $Q_i = S_{G_i}$. The rest of the notation is standard and can be found in any introductory book on (finite) group theory (e.g., [Asch] or [Go]).

1. PRELIMINARY RESULTS

1.1. *Minimal Parabolic Systems and Small Representations*

In this subsection we describe some of the major tools used in the classification of minimal parabolic systems. Most of the information is taken from [SW]. We start with a lemma that helps us to determine the structure of the kernels K_i.

LEMMA 1.1. *Let G be a group with a* (*semi*-)*classical parabolic system* $\{P_i | i \in I\}$.

(i) *Let $i \neq j \in I$ and let p be a prime. If p does not divide $|K_i K_j / K_j|$ and $|K_i K_j / K_i|$, then p does not divide $|K_i K_j|$.*

(ii) *If $P_i / B_{P_i} \cong \Sigma_3$ for all $i \in I$, then $B = S$ and K_i is a* 2-*group for all i.*

Proof. (i) is [S1, (1.14)]; (ii) is [S1, (1.15)] or [S2, (1.4)]. ∎

In the notation introduced above, it is straightforward to see that Z_i is elementary abelian and that $[Z_i, Q_i] = 1$. Moreover, if $C_S(Q_i) \leq Q_i$ then either Z_i is a nontrivial G_i/Q_i-module or $\Omega_1(Z(S)) = Z_i \trianglelefteq G_i$. By axiom (v) of Definition 1 the latter can occur for at most one $i \in I$. On the other hand, if $C_S(Q_i) \nleq Q_i$ in most cases the structure of G_i/Q_i implies that $G_i = C_{G_i}(Q_i) K_i$ which in turn often yields a contradiction to the action of G_{ij} on $Q_i K_j / K_j$ for some $j \neq i$. In the following we will therefore provide some tools which are useful for studying the structure of Z_i as a G_i-module.

DEFINITION 2. Let G be a group, V a faithful $GF(2)G$-module, $A \leq G$ an elementary abelian 2-group, and $t \in G$ an involution.

(i) If $|V\colon C_V(A)| \leq 2^i \cdot |A|$ for some $i \in \mathbb{N}_0$, then V is called an *F_i-module* and A is called an *offending subgroup*. If $i = 0$ we just say *F-module*.

(ii) If $|V\colon C_V(t)| \leq 2^{m_2(G)+i}$ for some $i \in \mathbb{N}_0$, then V is called an *$(SC + i)$-module*.

Definition 3. Let G be a group and G_1, G_2 proper subgroups of G such that $G = \langle G_1, G_2 \rangle$. The *coset graph* $\Gamma = \Gamma(G_1, G_2)$ is the graph with vertex set the right cosets of G_1 and G_2 in G, where two cosets $G_i g, G_j h$, $g, h \in G$, $i, j \in \{1, 2\}$, are incident iff they are distinct and $G_i g \cap G_j h \neq \emptyset$. For $\alpha, \beta \in \Gamma$ by $d(\alpha, \beta)$ we denote the usual distance metric on Γ and by $\Delta(\alpha) = \{\beta \in \Gamma \mid d(\alpha, \beta) = 1\}$ the neighborhood of $\alpha \in \Gamma$. If $\alpha = G_i g$, $g \in G$, we say α is conjugate to i and write $\alpha \sim i$.

The group G naturally acts on Γ by right multiplication and the proof of the following lemma is an easy exercise.

Lemma 1.2. (i) *Γ is bipartite and connected.*

(ii) *G acts faithfully on Γ iff there is no nontrivial normal subgroup of G contained in $G_1 \cap G_2$.*

(iii) *G acts edge-transitively on Γ.*

(iv) *If $\alpha = G_i g \in \Gamma$, then $G_\alpha = G_i^g$ is the stabilizer of α and G_α is transitive on $\Delta(\alpha)$.*

If we consider $\Gamma(G_1, G_2)$, where G_1 and G_2 are some parabolic subgroups of the group G we want to determine, then, in particular, $S \leq G_1 \cap G_2$ and $Z_1 \leq G_2$, $Z_2 \leq G_1$. Further it follows from axiom (v) of Definition 1 that the equivalent statements of Lemma 1.2(ii) hold. Thus there are $g, h \in G$ and $i, j \in \{1, 2\}$ such that $Z_1 \nleq G_i^g$, $Z_2 \nleq G_j^h$. The following definition will be useful (where $Z_\alpha = Z_i^g$ for $\alpha = G_i g$).

Definition 4. Let $\alpha, \beta \in \Gamma$ with $Z_\alpha \leq G_\beta$ but $Z_\alpha \nleq G_\gamma$ for some $\gamma \in \Delta(\beta)$ and $b = d(\alpha, \beta)$ minimal with respect to this property. Then (α, β) is called a *critical pair*.

The next three lemmas show that, if $Z_i \neq \Omega_1(Z(S))$, then in many cases Z_i turns out to be an F_1-module. Sometimes we also get some information about the offending subgroup. Since groups tend to possess not so many F_1-modules and often these modules are classified, this helps us to determine the action of G_1 on Z_1.

Lemma 1.3. *Let (α, β) be a critical pair in $\Gamma(G_1, G_2)$ and suppose that $C_S(Q_i) \leq Q_i$ for $i = 1, 2$.*

(i) *If $\alpha \sim j$ then $\Omega_1(Z(S)) \ntrianglelefteq G_j$.*

(ii) *If $\Omega_1(Z(S)) \ntrianglelefteq G_i$, for $i \sim \alpha, \beta$, then $[Z_\alpha, Z_\beta] \neq 1$ and Z_α is an F-module with offending subgroup $Z_\beta C_{G_\alpha}(Z_\alpha)/C_{G_\alpha}(Z_\alpha)$ or Z_β is an F-module with offending subgroup $Z_\alpha C_{G_\beta}(Z_\beta)/C_{G_\beta}(Z_\beta)$.*

Proof. Notice that by conjugation there always exists a critical pair $(1, \beta)$ or $(2, \beta)$ and by Lemma 1.2(iv) we can assume $d(i, \beta) = d(j, \beta) - 1$, where (j, β) is a critical pair and $\{i, j\} = \{1, 2\}$. So if $\Omega_1(Z(S)) \trianglelefteq G_j$ then $Z_j = \Omega_1(Z(S)) \leq Z_i$ and there is no critical pair (j, β), which proves (i).

If $[Z_\alpha, Z_\beta] = 1$, then $Z_\alpha \le C_{G_\beta}(Z_\beta) \le Q_\beta \le G_{\Delta(\beta)}$ in contradiction to the choice of (α, β). Now if $|Z_\alpha\colon Z_\alpha \cap K_\beta| \le |Z_\beta\colon Z_\beta \cap K_\alpha|$, then $|Z_\alpha\colon C_{Z_\alpha}(Z_\beta K_\alpha/K_\alpha)| \le |Z_\beta K_\alpha/K_\alpha|$ and Z_α is an F-module with offending subgroup as stated. Otherwise we get (ii) by interchanging the roles of α and β. ∎

The following is basically [SW, (2.1), (2.2)] but as the assumptions differ a bit we give a proof. Recall that for any p-group P the *Thompson subgroup* $J(P)$ of P is the group $J(P) = \langle A \mid A \in \mathscr{A}(P)\rangle$, where

$$\mathscr{A}(P) = \{A \le P \mid A \text{ is elementary abelian}, |A| \text{ maximal}\}.$$

Further, we set $\tilde{Z}(P) = \Omega_1(Z(J(P))$ and $\tilde{J}(P) = C_P(\tilde{Z}(P))$.

LEMMA 1.4. *Let $G = \langle G_1, G_2\rangle$. Set $B = G_1 \cap G_2$ and $K_i = B_{G_i}$, $i = 1, 2$. Assume that B_{G_1} is a 2-group, G_2 is finite, and that the following hold*:

(i) *There is $S \le G_1 \cap G_2$, $S \in Syl_2(G_2)$. Set $Z_i = \langle \Omega_1(Z(S))^{G_i}\rangle$, $i = 1, 2$. Then $Z_1 \le K_1$ and any 2-element in $G_1 \cap G_2$ centralizing Z_1 is in K_1.*

(ii) *$G_2/B_{G_2} \cong L_2(q)$, $q = 2^h$ for some h, and $Z_2 \le O_2(G_2) = S_{G_2}$.*

(iii) *$B_G = 1$.*

Then either there exists some $A \le S_{G_2}$ such that $[A, Z_1] \ne 1$, AK_1/K_1 is elementary abelian, and $|Z_1\colon C_{Z_1}(A)| \le q \cdot |A\colon A \cap K_1|$, or, for $\rho \in \langle K_1^{G_2}\rangle$, $\rho^{q+1} \in G_1 \cap G_2$, we have $\|[\rho, O_2(G_2)]\| = q^2$ and $\|[O_2(\langle S, \rho\rangle), \rho]\| = q^2$.

Proof. As $K_1 \cap K_2 \trianglelefteq K_2$ and K_1 is a 2-group, we have $K_1 \cap K_2 \le O_2(G_2)$ and $[K_1 \cap K_2, Z_2] = 1$. We consider the coset graph $\Gamma(G_1, G_2)$.

Let first $(1, \alpha)$ be a critical pair. If $\alpha \sim 1$ then, by (i), $[Z_1, Z_\alpha] \ne 1$. So we may assume $1 \ne |Z_1\colon C_{Z_1}(Z_\alpha)| \le |Z_\alpha\colon C_{Z_\alpha}(Z_1)| = |Z_\alpha\colon Z_\alpha \cap K_1|$. Hence Z_1 is even an F-module with offending subgroup $Z_\alpha K_1/K_1$. Let $(1, 2, 3, \ldots, \alpha)$ be a path of length b. Then $Z_\alpha \le K_3 \cap K_2 \le O_2(G_2)$. So $Z_\alpha \le S_{G_2}$.

Now assume $\alpha \sim 2$. Let $[Z_1, Z_\alpha] \ne 1$. As $Z_1 \le K_{\alpha-1}$ and $[Z_1 \cap K_\alpha, Z_\alpha] = 1$, we get $|Z_1\colon C_{Z_1}(Z_\alpha)| \le |K_{\alpha-1}\colon K_{\alpha-1} \cap K_\alpha| = q \le q \cdot |Z_\alpha\colon Z_\alpha \cap K_1|$, the assertion. So let $[Z_1, Z_\alpha] = 1$. Let $\alpha + 1 \in \Delta(\alpha)$ with $d(1, \alpha + 1) = b + 1$. Then $|Z_1\colon Z_1 \cap G_{\alpha+1}| \le q$. Similarly, we have $|Z_{\alpha+1}\colon Z_{\alpha+1} \cap G_1| \le q$. Let $|Z_1 \cap G_{\alpha+1}\colon C_{Z_1 \cap G_{\alpha+1}}(Z_{\alpha+1})| \le |Z_{\alpha+1} \cap G_1\colon C_{Z_{\alpha+1} \cap G_1}(Z_1)|$. Then either $A = Z_{\alpha+1} \cap G_1$ has the desired properties or we have $[Z_1 \cap G_{\alpha+1}, Z_{\alpha+1}] = [Z_{\alpha+1} \cap G_1, Z_1] = 1$. So assume the latter. Now we have $Z_{\alpha+1} \cap G_1 \trianglelefteq \langle K_{\alpha+1}, Z_1\rangle = Y$ and $G_\alpha = Y(G_\alpha \cap G_{\alpha-1})$. We have $K_{\alpha+1} \cap K_\alpha \le C_{K_\alpha}(Z_{\alpha+1} \cap G_1)$. If $C_{K_\alpha}(Z_{\alpha+1} \cap G_1) > K_{\alpha+1} \cap K_\alpha$, then, for $t \in C_{K_\alpha}(Z_{\alpha+1} \cap G_1)$, $t \notin K_{\alpha+1}$, we have $|Z_{\alpha+1}\colon C_{Z_{\alpha+1}}(t)| \le q$. So by symmetry between 1 and $\alpha+1$ we may assume that $K_\alpha \cap K_{\alpha+1}$ is normalized by Y. As $[K_{\alpha+1}, K_\alpha] \leq K_\alpha \cap K_{\alpha+1}$ and $Y \leq \langle K_{\alpha+1}^Y \rangle$, we see that $[Y, K_\alpha] \leq K_\alpha \cap K_{\alpha+1}$. As $YK_\alpha/K_\alpha \cong L_2(q)$, $|K_{\alpha+1}: K_{\alpha+1} \cap K_\alpha| = q$, and there are involutions in $K_{\alpha+1} \setminus K_\alpha$, we see that $K_{\alpha+1} \in Syl_2(Y)$. Furthermore, $|Y: Y \cap G_{\alpha+1}| = q+1$ and $Y \cap G_{\alpha+1}$ normalizes $K_{\alpha+1}$. This implies that $Y/K_\alpha \cap K_{\alpha+1} \cong L_2(q)$. Let C be a characteristic subgroup of $K_{\alpha+1}$ and $C \trianglelefteq Y$. Then $C \trianglelefteq \langle G_{\alpha+1}, Y \rangle = G$. So $C = 1$. Now by [Bau] we get $\llbracket \tau, K_\alpha \cap K_{\alpha+1} \rrbracket = q^2$ for $\tau \in Y$, $o(\tau) = q+1$. Then for a conjugate ρ of τ in G_2 with $\langle S, \rho \rangle / O_2(\langle S, \rho \rangle) \cong L_2(q)$, We have $\llbracket O_2(\langle S, \rho \rangle), \rho \rrbracket = q$, the assertion.

Finally, let $(2, \alpha)$ be a critical pair and $\alpha \sim 2$. Then, in particular, $K_{\alpha+1} \nleq K_\alpha$. Suppose Z_1 is not an F-module with offending subgroup in K_2. Then $J(K_1K_2) \leq K_1$ and $J(K_1K_2) = J(K_1)$. Let $Y = \langle K_1^{G_2} \rangle K_2$. Then $Y/K_2 \cong L_2(q)$. Further $J(K_1K_2) \nleq K_2$. By [Bau] (see also [Cher, (2.1)]) we have $\tilde{J}(K_1K_2) \in Syl_2(\langle \tilde{J}(K_1K_2)^Y \rangle)$ and $\langle \tilde{J}(K_1K_2)^Y \rangle K_2/K_2 \cong L_2(q)$. Since $Z_1 \leq \tilde{Z}(K_1K_2)$ we see that $\tilde{J}(K_1K_2) \leq K_1$. In particular, $\tilde{J}(K_1K_2) = \tilde{J}(K_1) \trianglelefteq G_1$. As $G = \langle G_1, Y \rangle$ we have that no nontrivial characteristic subgroup of K_1 is normal in $X = \langle \tilde{J}(K_1K_2)^Y \rangle$. By [Bau] we get $\llbracket K_1K_2, \rho \rrbracket = q^2$ for $\rho \in X$, $o(\rho) = q+1$. Now we see as before that for $P = \langle S, \rho \rangle$ we have $\llbracket O_2(P), \rho \rrbracket = q^2$, the assertion. ∎

LEMMA 1.5. *Let $\{P_1, \ldots, P_n\}$ be a semiclassical parabolic system for a group G such that the assumptions of Lemma* 1.4 *are satisfied with $G_2 = P_1$ and $G_1 = \langle P_2, \ldots, P_n \rangle$. Suppose $P_1/B_{P_1} \cong \Sigma_3$, the diagram of P_{12} is connected, and $C_S(S_{P_{12}}) \leq S_{P_{12}}$. Then Z_1 is an F_1-module for $G_1/C_{G_1}(Z_1)$ with offending subgroup in S_{P_1}.*

Proof. Suppose false. Then by (1.4) we have $\llbracket \rho, S_{P_1} \rrbracket = 4$ for $\rho \in P_1$, $o(\rho) = 3$ (notice $q = 2$ by assumption). As $P_{12}/B_{P_{12}}$ is a rank-2 Lie group or $3\Sigma_6$ resp. $3A_6$, we see $[\rho, S_{P_1}] \cap B_{P_{12}} = 1$. Hence $[\rho, S_{P_{12}}] = 1$. But then 2 divides $|C_{P_{12}}(S_{P_{12}})S_{P_{12}}/S_{P_{12}}|$, a contradiction. ∎

1.2. *Some Results about Modules*

In this subsection we collect some representation results, most of them concerning the group $Sp_6(2)$. The first lemma, which was proved in [Gil, (3.4.2)], will often enable us to reduce consideration of modules for $3^7Sp_6(2)$ to that of modules for $Sp_6(2)$.

LEMMA 1.6. *Let $G = 3^7Sp_6(2)$ and $0 \neq V$ an irreducible $GF(2)G$-module. If $|V: C_V(t)| \leq 2^7$ for some involution $t \in G$ (i.e., V is an $(SC+1)$-module), then $[O_3(G), V] = 0$.*

LEMMA 1.7. *Let $G = Sp_{2n}(2)$, $n \geq 3$, and let V be a $GF(2)G$-module.*

(i) *If V is an F_1-module and V is irreducible, then V is the natural module, or $G = Sp_6(2)$ resp. $Sp_8(2)$ and V is the* 8-*resp.* 16-*dimensional spin module, and in the last case V is not an F-module.*

(ii) *If $G = Sp_6(2)$ and V is a (possibly reducible) F_1-module, then V involves as nontrivial modules at most two natural modules or one spin module.*

(iii) *If $G = Sp_6(2)$, V is irreducible and V is an SC-module but not an F_1-module, then $V = M(\lambda_2)$ (in the notation of* [Co]).

Proof. (i) and (iii) are taken from [S1, (1.1)]; (ii) is shown in [Gil, (3.5)]. ∎

LEMMA 1.8. *Let $G = Sp_{2n}(2)$ and let V be an irreducible $GF(2)G$-module. Let $P_1, \ldots, P_n$ be the minimal parabolics of G corresponding to the diagram $\overset{1}{\circ}\!=\!\overset{2}{\circ}\!-\!\overset{3}{\circ}\ \cdots\ \overset{n-1}{\circ}\!-\!\overset{n}{\circ}$ and $S \in Syl_2(P_i)$ for all i.*

(i) *If V is the natural module, then $N_G(C_V(S)) = \langle P_1, \ldots, P_{n-1} \rangle$.*

(ii) *If V is the spin module, then $N_G(C_V(S)) = \langle P_2, \ldots, P_n \rangle$.*

Proof. [S1, (1.3)]. ∎

In the following lemmas, we classify involutions as in [AS] based on [Suz].

LEMMA 1.9. *Let $G = Sp_6(2)$, V the natural $GF(2)G$-module, $G_1 \leq G$ the stabilizer of a nontrivial vector of V, and $A \leq O_2(G_1)$. Then*:

(i) *G does not contain a* 4-*group of transvections.*

(ii) *$|V: C_V(A)| \geq |A|$, and if A acts quadratically on V, then $|A| \leq 8$.*

(iii) *If $|A| \geq 8$ then A contains an element of type c_2.*

(iv) *Let $t \in G$ such that t induces a transvection on V. Suppose W is a seven-dimensional $GF(2)G$-module with $W/C_W(G) \cong V$. Then $|[W, t]| = 2$ and $C_{W/C_W(G)}(t) = C_W(t)/C_W(G)$.*

Proof. (i)–(iii) are shown in [Gil, (2.4), (2.6), (3.2.2), and (3.2.3)]. In (iv) we may assume $G_1 = C_G(t)$. We have $|[W, t]| \leq |[V, t]| \cdot |C_W(G)| = 4$. So $|C_W(t)| \geq 2^5$ and $|C_{W/C_W(G)}(t): C_W(t)C_W(G)/C_W(G)| \leq 2$. Now (iv) follows from the fact that $C_V(t)$ does not contain a G_1-submodule of index 2. ∎

LEMMA 1.10. *Let G, V, and G_1 be as in Lemma* 1.9. *Let W be the spin module for G and $A \leq G$ an elementary abelian subgroup with $|A| = 4$.*

(i) *If all involutions in A are of type a_2, then $|[V, A]| \geq 2^3$ and $|[W, A]| \geq 2^3$.*

(ii) *If $A \leq O_2(G_1)$, Then $|[V, A]| \cdot |[W, A]| \geq 2^6$.*

Proof. Let $A = \langle s, t\rangle$. If s is of type a_2, then $\|[U, s]\| = 2^2$ for $U \in \{V, W\}$. Moreover, $C_G(s) \cong 2^7(\Sigma_3 \times \Sigma_3)$ acts on $[U, s]$ and $C_G(s)$ is a maximal subgroup of G with center $\langle s\rangle$. Therefore, if t is also of type a_2, then $G = \langle C_G(s), C_G(t)\rangle$. This implies $[U, s] \neq [U, t]$ and (i) follows.

In (ii) notice that $O_2(G_1)$ only contains involutions of type b_1, c_2, and a_2. If all involutions in A are of type a_2, the assertion follows from (i). Otherwise, we get $\|[W, A]\| = 2^4$ from [S1, (1.8)] and $\|[V, A]\| \geq 2^2$ from Lemma 1.9(i). ∎

LEMMA 1.11. *Let $G = Sp_6(2)$, $V = M(\lambda_2)$, and $t \in G$ an involution. Then $\|[V, t]\| \geq 2^4, 2^5, 2^6$ if it is of type b_1, α_2 resp. c_2.*

Proof. Let $t \in G$ be of type b_1 and $G_1 = C_G(t)$. Then $[V, t]$ involves a nontrivial module for $G_1/O_2(G_1) \cong \Sigma_6$. Let $0 \neq V_1 \leq [V, t]$ be an irreducible submodule. Then $\|[V, t]\| \geq |V_1| = 2^4$ by [Tim, (2.17)].

Since $V/C_V(t) \cong [V, t]$ as a G_1-module, there is $V_2 \leq V$ such that $V/V_2 \cong V_1$. Now $\dim V_2/V_1 = 14 - 2 \cdot 4 = 6$ and V_2/V_1 must involve a further nontrivial Σ_6-module. (Otherwise, we would get the contraction $\dim C_V(S) > 1$ for $S \in Syl_2(G)$.

Choose $s \in O_2(G_1)$ of type a_2. Then st is of type c_2 and $P = G_1 \cap C_G(s)$ is a minimal parabolic subgroup of G corresponding to an end node of the Dynkin diagram. Hence $[P, C_V(S)] = 0$ for a suitable Sylow-2-subgroup $S \leq P$ by the definition of the module $M(\lambda_2)$. Since $C_V(S) \leq V_1$, this implies that $V_1 \cong O_2(G_1)/\langle t\rangle$ as a $G_1/O_2(G_1)$-module. Calculation in the group $Sp_6(2)$ shows that s is conjugate in G to an element $\tilde{s} \in G_1 \setminus O_2(G_1)$ such that $\|[V_1, \tilde{s}]\| = \|[V/V_2, \tilde{s}]\| = 2^2$. Hence $\|[V, s]\| \geq 2^5$. Also, st is conjugate to an element in $G_1' \setminus O_2(G_1)$; i.e., it corresponds to an element in A_6. This implies $\|[V, st]\| \geq 2^6$. ∎

Remark. In [S1, (1.17)(b)(i)] it was stated that $|V\colon C_V(t)| \geq 2^6$ holds for any involution $t \in G$. But, considering the module $M(\lambda_1)$ for the group $F_4(2)$, which involves a section isomorphic to V, one can see that $\|[V, t]\| = 2^4$ if t is of type b_1. (It can even be shown independently by considering that module that we actually always have equality in Lemma 1.11. But since we do not need this fact for our applications, we did not include it as it would make the proof of Lemma 1.11 too long.)

LEMMA 1.12. *Let $G = Sp_6(2)$ and let V be a $GF(2)G$-module such that $W = [V, G]$ is the spin module. Then $V = W \oplus C_V(G)$.*

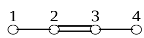
Proof. It suffices to consider the case $|V\colon W| = 2$. Let $K \leq G$ with $K \cong O_6^+(2) \cong \Sigma_8$, $A_8 \cong H \leq K$. Let G_1 be the stabilizer of a point in the natural representation of G which is nonsingular with respect to the quadratic form stabilized by K. Then $H_1 = H \cap G_1 \cong \Sigma_6$. (This can easily be calculated taking into account that the natural $GF(2)K$-module is isomorphic to a section of the permutation module for Σ_8.) Thus H_1 contains a Sylow-3-subgroup of G_1 and there is $\rho \in H_1$ such that ρ acts fixed point freely on W. Since $1, 4, 6$ are the dimensions less than or equal to 8 of the irreducible A_8-modules, as an H-module W must involve two four-dimensional modules. Now $V = W \oplus C_V(H)$ by [S1, (1.7)].

As $\rho \in H_1$ acts fixed point freely on W, we have $C_V(H) = C_V(H_1)$. Let $\langle t \rangle = Z(G_1)$. Then $C_V(H_1)^t = C_V(H_1)$. On the other hand, $C_V(H)$ is invariant under $N_G(H) = K$ and hence under $\langle t, K \rangle = G$. (By [A] K is a maximal subgroup of G.) ∎

LEMMA 1.13. *Let $G = Sp_6(2)$, $G_1 \leq G$ the stabilizer of a vector in the natural module, and $K_1 = O_2(G_1)$. Let V be the spin module and $0 \neq v \in V$. Then $|K_1 \colon C_{K_1}(v)| \leq 8$.*

Proof. Let $V_1 = C_V(K_1) = [V, K_1]$. Then V_1 and V/V_1 are irreducible modules for $G_1/K_1 \cong \Sigma_6$ of dimension 4. Hence G_1 is transitive on $V_1^{\#}$ and $(V/V_1)^{\#}$. Let $v \in V \setminus V_1$. As $[v, K_1] \subseteq V_1$, we have $|K_1 \colon C_{K_1}(v)| \leq 16$, and we have equality iff v is conjugate to all elements in $v + V_1$. But then, as V_1 is not invariant under G, G would be transitive on $V^{\#}$, a contradiction to $C_G(C_V(S)) \cong 2^6 L_3(2)$ for $S \in Syl_2(G)$. ∎

2. THE GLOBAL STRUCTURE OF THE MAXIMAL PARABOLICS

Let $H = F_4(2)$. Then H has the Dynkin diagram  and for the two maximal parabolic subgroups corresponding to the end nodes we have $H_1 \cong H_4 \cong 2^{1+6+8} Sp_6(2)$ and $H_1 \cap H_4 / O_2(H_1 \cap H_4) \cong Sp_4(2) \cong \Sigma_6$. Moreover, for $i = 1, 4$, $O_2(H_i)$ is the direct product of an extraspecial group of order 2^{1+8} and an elementary abelian group of order 2^6 and $H_i/O_2(H_i)$ acts faithfully and indecomposably on $Z(O_2(H_i))$ and on $O_2(H_i)/Z(H_i)$. The main goal of this section is to show that G_1 and G_4 contain normal subgroups of order 3^7 such that the same is true for $G_i/O_3(G_i)$.

LEMMA 2.1. *Let G be a group of type $\tilde{F}_4$. Then $B = S$ and $K_{ij} = O_2(G_{ij})$ for all $1 \leq i, j \leq 4$ (including the case $i = j$).*

Proof. This follows from Lemma 1.1. ∎

LEMMA 2.2. *Let G be a group of type $\tilde{F}_4$. Suppose $C_S(K_i) \leq K_i$ for $i = 1, 4$ and $\Omega_1(Z(S)) \trianglelefteq G_1$. Let $(4, \alpha)$ be a critical pair in $\Gamma(G_1, G_4)$. Then $\alpha \sim 1$.*

Proof. Suppose to the contrary that $\alpha \sim 4$. Then, by Lemma 1.3, Z_4 is an F-module with offending subgroup contained in $O_2(G_{14}/K_4) = K_{14}/K_4$ and by Lemma 1.6, $[Z_4, N_4] = 1$. Since $[\Omega_1(Z(S)), G_{14}] = 1$, by Lemmas 1.7 and 1.8, Z_4 involves the natural module. Now Lemma 1.9(ii) implies that Z_4 cannot involve another nontrivial module; therefore Z_4 is the natural module, $|Z_4| = 2^6$, and $|\Omega_1(Z(S))| = |Z_1| = 2$.

Let $\tilde{Z}_4 = [Z_4, K_{14}]$, $\tilde{V}_1 = \langle \tilde{Z}_4^{G_1} \rangle$, and $V_1 = \langle Z_4^{G_1} \rangle$. Then $\tilde{Z}_4 \trianglelefteq G_{14}$ and $|Z_4 \colon \tilde{Z}_4| = 2$. Moreover, G_1 acts on V_1 and $\tilde{V}_1$ and we are going to derive contradictions from these actions.

First of all, since $\alpha \sim 4$, we have $Z_4 \leq K_1$, so $V_1 \leq K_1$, too. Further, the action of K_{14}/K_4 yields

$$\tilde{V}_1' \leq \left[\tilde{V}_1, K_1\right] \leq \left[\tilde{Z}_4, K_1\right]^{G_1} \leq Z_1^{G_1} = Z_1 \tag{2.1}$$

and

$$V_1' \leq [V_1, K_1] \leq [Z_4, K_1]^{G_1} \leq \langle \tilde{Z}_4^{G_1} \rangle = \tilde{V}_1. \tag{2.2}$$

In particular, we can regard $\tilde{V}_1/Z_1$ and $V_1/\tilde{V}_1$ as $GF(2)$-modules for G_1/K_1. As G_{14}/N_{14} acts nontrivially on $\tilde{Z}_4/Z_1$, $\tilde{V}_1/Z_1$ must involve a nontrivial (G_1/K_1)-module, and since $[Z_4, K_1] \neq [\tilde{Z}_4, K_1]$ we have $Z_4 \not\leq \tilde{V}_1$ and $V_1 > \tilde{V}_1$. We can even show the following two assertions:

If $\tilde{V}_1$ is an SC-module, then $\tilde{V}_1$ involves the spin module, and if $\tilde{V}_1$ involves only one nontrivial module, then $|\tilde{V}_1| = 2^9$. (Notice that by the assumption "SC-module" and, by Lemma 1.6, *$\tilde{V}_1$ is a module for $G_1/N_1 \cong Sp_6(2)$.)* (2.3)

If V_1 is an $(SC + 1)$-module and $|K_1 \colon K_1 \cap K_4| = 2^5$, then $\tilde{V}_1$ involves the spin module and $V_1/\tilde{V}_1$ involves the natural module. (2.4)

In order to prove (2.3) we set $T = \langle Z_1^{P_1} \rangle$ and choose $V \leq \tilde{V}_1$ maximal subject to $T \not\leq V$ and $V \trianglelefteq G_1$. Then $\tilde{V}_1/V$ is a nontrivial irreducible module with $TV/V \leq C_{\tilde{V}_1/V}(S/K_1)$ and $[T, P_2] \not\leq V$. So $\tilde{V}_1/V$ can neither be the natural module nor $M(\lambda_2)$ and hence has to be the spin module by Lemma 1.7. If V/Z_1 is a trivial (G_1/N_1)-module, then, by the dual of Lemma 1.12, V_1/Z_1 splits over V/Z_1; i.e., we have

$$\tilde{V}_1/Z_1 = V/Z_1 \oplus \left[\tilde{V}_1, G_1\right] Z_1/Z_1,$$

with $|[\tilde{V}_1, G_1]Z_1| = 2^9$. But from the action of $G_{14} \leq G_4$ on Z_4, we know that $\tilde{Z}_4 = [\tilde{Z}_4, G_{14}]$. Thus $\tilde{V}_1 = \langle [\tilde{Z}_4, G_{14}]^{G_1} \rangle \leq [\tilde{V}_1, G_1]^{G_1} = [\tilde{V}_1, G_1]$ and $V = Z_1$; i.e., (2.3) is shown.

Suppose G_1/N_1 acts trivially on $V_1/\tilde{V}_1$. Then $Z_4\tilde{V}_1 \trianglelefteq G_1$. Hence also $[Z_4\tilde{V}_1, K_1] \trianglelefteq G_1$. But, under the assumptions of (2.4), we have $K_1K_4/K_4 = O_2(G_{14}/K_4)$ and the action of $Sp_6(2)$ on the natural module implies that $\tilde{Z}_4 = [Z_4, K_1] = [Z_4\tilde{V}_1, K_1] \trianglelefteq G_1$. This is impossible because $|\tilde{Z}_4| = 2^5$. Therefore $V_1/\tilde{V}_1$ must be a nontrivial SC-module with $Z_4\tilde{V}_1/\tilde{V}_1 \leq C_{V_1/\tilde{V}_1}(S/K_1)$ and $[Z_4, G_{14}] \leq \tilde{Z}_4 \leq \tilde{V}_1$, and hence involves the natural module by Lemmas 1.7 and 1.8. Now $\tilde{V}_1$ is still an SC-module and the rest of (2.4) follows from (2.3).

Let α be chosen such that $d(1, \alpha) = d(4, \alpha) - 1$ and let $\alpha + 1 \in \Delta(\alpha)$ with $Z_4 \nleq G_{\alpha+1}$. Let $b = d(4, \alpha)$. We are going to derive contradictions in each of the cases $b = 2$ and $b > 2$. First suppose:

Case 1: $b = 2$, i.e., $\alpha \in \Delta(1)$. We will distinguish the subcases $\tilde{V}_1 \nleq K_4$ and $\tilde{V}_1 \leq K_4$.

Case 1.1: $\tilde{V}_1 \nleq K_4$. From $Z_4 \leq K_1$ and (2.1) we get $[Z_4, \tilde{V}_1] \leq Z_1$, which means that $\tilde{V}_1$ induces transvections on Z_4. By Lemma 1.9(i) there is no 4-group of transvections in $Sp_6(2)$. Hence

$$|\tilde{V}_1 : \tilde{V}_1 \cap K_4| = 2. \tag{2.5}$$

Furthermore, $[\tilde{V}_1, \tilde{Z}_4] = 1$, so $\tilde{V}_1' = [\tilde{V}_1, \langle \tilde{Z}_4^{G_1} \rangle] = 1$ and $\tilde{V}_1$ is elementary abelian.

As $Z_4 = \langle Z_\beta \mid \beta \in \Delta(4) \rangle$, there exists $\beta \in \Delta(4)$ with $[\tilde{V}_1, Z_\beta] \neq 1$. Thus $\tilde{V}_1 \nleq G_\beta, \tilde{V}_1 \cap Z_\beta = 1$ and

$$\left[\tilde{V}_1 \cap K_\beta, \tilde{V}_\beta \cap G_1\right] \leq \left[\tilde{V}_1, G_1\right] \cap \left[\tilde{V}_\beta, K_\beta\right] \leq \tilde{V}_1 \cup Z_\beta = 1. \tag{2.6}$$

As $\tilde{V}_1 \cap K_4 \leq G_\beta$, we deduce from (2.5) and (2.6) that

$$\begin{aligned} \left|\tilde{V}_1 : C_{\tilde{V}_1}\left(\tilde{V}_\beta \cap G_1\right)\right| &\leq |\tilde{V}_1 : \tilde{V}_1 \cap K_\beta| \\ &= |\tilde{V}_1 : \tilde{V}_1 \cap G_\beta| \cdot |\tilde{V}_1 \cap G_\beta : \tilde{V}_1 \cap K_\beta| \\ &= 2 \cdot |\tilde{V}_1 \cap G_\beta : \tilde{V}_1 \cap K_\beta|. \end{aligned} \tag{2.7}$$

As β is conjugate to 1 in G_4, we have $|\tilde{V}_\beta : \tilde{V}_\beta \cap K_4| = 2$, too, and inequality (2.7) also holds, if we interchange 1 and β. So if $\tilde{V}_\beta \cap K_4 \nleq K_1$ we may assume $|\tilde{V}_1 \cap G_\beta : \tilde{V}_1 \cap K_\beta| \leq |\tilde{V}_\beta \cap G_1 : \tilde{V}_\beta \cap K_1|$, which means that $\tilde{V}_1$ is an F_1-module with offending subgroup $(\tilde{V}_\beta \cap G_1)K_1/K_1$. Now we get a contradiction to (2.3) because $\tilde{V}_1 \cap K_4$ is a G_{14}-submodule of index 2 in $\tilde{V}_1$ and the spin module does not possess such a submodule.

So we have $\tilde{V}_\beta \cap K_4 = \tilde{V}_\beta \cap K_1$, which yields $[\tilde{V}_1, \tilde{V}_\beta \cap K_4] \le Z_1 \le Z_4$ and $(\tilde{V}_\beta \cap K_4)Z_4 \trianglelefteq \langle G_4 \cap G_\beta, \tilde{V}_1 \rangle = G_4$. Consequently, $W = [K_4, (\tilde{V}_\beta \cap K_4)Z_4] = [K_4, \tilde{V}_\beta \cap K_4] \trianglelefteq G_4$. Now $K_4 \not\le K_\beta$ and by (2.3), K_4 does not induce transvections on $\tilde{V}_\beta$. In particular, $W \ne 1$ by (2.5). So $W \cap \Omega_1(Z(S)) \ne 1$ and we get the contradiction $Z_4 \le W \le \tilde{V}_\beta$.

Case 1.2: $\tilde{V}_1 \le K_4$, in particular, $[\tilde{V}_1, Z_4] = 1$. By conjugation in G_1, for all $\delta \in \Delta(1)$ we get $[\tilde{V}_1, Z_\delta] = 1$. Hence $\tilde{V}_1 \le Z(V_1) \le K_\delta$ and $|Z_\delta \colon Z_\delta \cap Z(V_1)| = 2$. Moreover, $\tilde{Z}_4 \le K_\alpha \le G_{\alpha+1}$ and $V_1' = [Z_4, Z_\alpha] = Z_1$. Thus V_1 induces a transvection on Z_α and

$$|V_1 \colon V_1 \cap K_\alpha| = 2 \tag{2.8}$$

by Lemma 1.9(i). By symmetry, we have $|V_{\alpha+1} \colon V_{\alpha+1} \cap K_\alpha| = 2$ and $\tilde{V}_{\alpha+1} \le K_\alpha \le G_1$. We are going to discuss the cases $\tilde{V}_{\alpha+1} \le K_1$ and $\tilde{V}_{\alpha+1} \not\le K_1$ separately.

Suppose first that $\tilde{V}_{\alpha+1} \le K_1 \le G_4$. Then

$$\left[\tilde{V}_{\alpha+1}, Z_4 \cap G_{\alpha+1}\right] = \left[\tilde{V}_{\alpha+1}, \tilde{Z}_4\right] \le \left[K_1, \tilde{Z}_4\right] \le Z_1$$

and (2.3) implies that $\tilde{Z}_4$ acts trivially on $\tilde{V}_{\alpha+1}/Z_{\alpha+1}$, i.e., $\tilde{Z}_4 \le K_{\alpha+1}$. Now $[\tilde{Z}_4, \tilde{V}_{\alpha+1}] \le Z_4 \cap Z_{\alpha+1} = 1$. As $Z_4 \not\le G_{\alpha+1}$, we have $[\tilde{V}_{\alpha+1}, Z_4] \ne 1$. So $\tilde{V}_{\alpha+1}$ induces a transvection on Z_4. This implies that $\tilde{V}_{\alpha+1}K_4/K_4 = Z_\alpha K_4/K_4$ and $[Z_4, \tilde{V}_{\alpha+1}] = [Z_4, Z_\alpha] \le Z_\alpha$. Therefore we have $\tilde{V}_{\alpha+1}Z_\alpha \trianglelefteq \langle Z_4, G_\alpha \cap G_{\alpha+1} \rangle = G_\alpha$, and hence also $[\tilde{V}_{\alpha+1}Z_\alpha, K_\alpha] \trianglelefteq G_\alpha$. But $[\tilde{V}_{\alpha+1}Z_\alpha, K_\alpha] = [\tilde{V}_{\alpha+1}, K_\alpha] \ne 1$, so $Z_{\alpha+1} \le [\tilde{V}_{\alpha+1}Z_\alpha, K_\alpha]$ and we get the contradiction

$$Z_\alpha = \langle Z_{\alpha+1}^{G_\alpha} \rangle \le \left[\tilde{V}_{\alpha+1}Z_\alpha, K_\alpha\right]^{G_\alpha} = \left[\tilde{V}_{\alpha+1}Z_\alpha, K_\alpha\right] \le \tilde{V}_{\alpha+1}.$$

It remains to consider the case $\tilde{V}_{\alpha+1} \not\le K_1$. First, we show that we may assume $V_{\alpha+1} \not\le G_1$. Indeed, as $1 \ne [Z_4, Z_\alpha]$, we have $[Z_4, x] \ne 1$ for $x \in Z_\alpha \setminus \tilde{Z}_\alpha$. Let τ be the involution in $Z((G_1 \cap G_\alpha)N_\alpha/N_\alpha)$ and $\sigma \in G_\alpha/N_\alpha$ an element of order 3 which is inverted by τ and centralizes a subgroup isomorphic to Σ_6 in G_α/N_α. Then $\langle Z_1, Z_1^\sigma \rangle$ is a hyperbolic plane in Z_α and $Z_1^\sigma \not\le \tilde{Z}_\alpha$. Take $\alpha + 1 = 1^\sigma$. Then

$$\begin{aligned}[V_{\alpha+1}, Z_1] &= [V_1^\sigma, Z_1] = \left[V_1, Z_1^{\sigma^{-1}}\right]^\sigma = [V_1, Z_1^{\tau\sigma\tau}]^\sigma \\ &= \left[V_1^\tau, Z_1^{\sigma\tau}\right]^\sigma = [V_1, Z_{\alpha+1}]^{\tau\sigma} \ne 1,\end{aligned}$$

which yields the assertion. In particular, Z_1 is not centralized by $V_{\alpha+1}$, and since, as stated previously, $\tilde{V}_{\alpha+1} \le Z(V_{\alpha+1})$, we have $Z_1 \not\le \tilde{V}_{\alpha+1}$.

Choose $t \in \tilde{V}_{\alpha+1} \setminus K_1$. Then by (2.8) and (2.1)

$$|V_1 \colon C_{V_1}(t)| \leq |V_1 \colon V_1 \cap K_\alpha| \cdot |V_1 \cap K_\alpha : V_1 \cap K_{\alpha+1}| \cdot |[K_{\alpha+1}, t]|$$
$$\leq 2 \cdot 2^5 \cdot 2 = 2^7.$$

As by (2.3), $|\tilde{V}_1 \colon C_{\tilde{V}_1}(\tilde{V}_{\alpha+1})| > 2$, we deduce from $\tilde{V}_1 \leq K_\alpha$ and

$$\left[\tilde{V}_1 \cap K_{\alpha+1}, \tilde{V}_{\alpha+1}\right] \leq \tilde{V}_1 \cap Z_{\alpha+1} = 1 \tag{2.9}$$

that $|K_1 \colon K_1 \cap K_4| = |K_\alpha \colon K_\alpha \cap K_{\alpha+1}| \geq |\tilde{V}_1 \colon \tilde{V}_1 \cap K_{\alpha+1} \geq 4$. Hence $|K_1 \colon K_1 \cap K_4| = 2^5$ and by (2.4) $\tilde{V}_1$ involves the spin module and $V_1/\tilde{V}_1$ the natural module.

Since $|\tilde{V}_1 K_{\alpha+1}/K_{\alpha+1}| \geq 4$, now we get $|\tilde{V}_{\alpha+1} \colon C_{\tilde{V}_{\alpha+1}}(\tilde{V}_1)| \geq 8$. But interchanging the roles of 1 and $\alpha + 1$ in (2.9) and taking into account that $Z_1 \nleq \tilde{V}_{\alpha+1}$, we see that $C_{\tilde{V}_{\alpha+1}}(\tilde{V}_1) = \tilde{V}_{\alpha+1} \cap K_1$. So $|\tilde{V}_{\alpha+1} K_1/K_1| \geq 8$ and, by Lemma 1.9(iii), $\tilde{V}_{\alpha+1}$ contains an element t of type c_2. Now, if $\tilde{V}_1$ involved more than one nontrivial module, we would get $|\tilde{V}_1 \colon C_{\tilde{V}_1}(t)| \geq 2^6$, which is impossible. So by (2.3), $|\tilde{V}_1| = 2^9$ and $\tilde{V}_1/Z_1$ is the spin module. Further $\|[\tilde{V}_1/Z_1, t]\| = 2^4$ and $[\tilde{V}_1, t] \leq \tilde{V}_{\alpha+1}$. But $Z_1 \nleq \tilde{V}_{\alpha+1}$. Thus $[\tilde{V}_1, t] \cap Z_1 = 1$ and $\|[\tilde{V}_1, t]\| = 2^4$. On the other hand, Lemma 1.13 yields $|\tilde{V}_1 \colon C_{\tilde{V}_1}(tZ_{\alpha+1})| \leq 8$, so t must be conjugate in $\tilde{V}_1$ to tz where $\langle z \rangle = Z_{\alpha+1}$, and we get the contradiction $Z_{\alpha+1} \leq [t, \tilde{V}_1] \leq \tilde{V}_1$.

Now consider

Case 2: $b > 2$. In this case $[Z_4, Z_\beta] = 1$ for all $\beta \in \Delta(1)$. In particular, $V_1 = \langle Z_\beta \mid \beta \in \Delta(1) \rangle \leq K_4$ and V_1 is elementary abelian.

Furthermore, $1 \neq Z_\alpha K_4/K_4 \leq O_2((G_1 \cap G_4)N_4/N_4)$ and from the action of $Sp_6(2)$ on the natural module we can see that $Z_1 \leq [Z_4, Z_\alpha] \leq Z_\alpha \leq V_{\alpha+1}$. Hence $V_{\alpha+1} \leq C_{G_\beta}(Z_1) = G_\beta \cap G_1$ for $\beta \in \Delta(1)$ with $d(\beta, \alpha) = b - 2$.

The action of $Sp_6(2)$ on the natural module also yields that $[Z_\alpha, t] \neq 1$ for $t \in Z_4 \setminus \tilde{Z}_4$, which implies that $Z_4 \cap G_{\alpha+1} = Z_4 \cap K_\alpha \leq \tilde{Z}_4$. Again we will distinguish two subcases.

Case 2.1. Suppose $\tilde{V}_{\alpha+1} \leq K_1 \leq G_4$. Then we have

$$\left[\tilde{V}_{\alpha+1}, Z_4 \cap G_{\alpha+1}\right] \leq \left[K_1, \tilde{Z}_4\right] \leq Z_1.$$

But by (2.3) there are no transvections on $\tilde{V}_{\alpha+1}$, so $Z_4 \cap G_{\alpha+1} \leq K_{\alpha+1}$. This yields $[Z_4 \cap G_{\alpha+1}, \tilde{V}_{\alpha+1}] \leq Z_4 \cap Z_{\alpha+1} = 1$ and

$$C_{Z_4}\left(\tilde{V}_{\alpha+1} K_4/K_4\right) = C_{Z_4}(Z_\alpha K_4/K_4) = Z_4 \cap K_\alpha. \tag{2.10}$$

Since $[\tilde{Z}_4, Z_\alpha] \leq Z_1$ and there is no 4-group of transvections on Z_α, we have $|\tilde{Z}_4 \colon \tilde{Z}_4 \cap K_\alpha| \leq 2$ and $|Z_4 \colon Z_4 \cap K_\alpha| \leq 4$. Suppose $s \in \tilde{V}_{\alpha+1} \setminus K_4$ does not induce a transvection on Z_4. Then by (2.10) $|Z_4 \colon Z_4 \cap K_\alpha| = 4$. Now, by Lemma 1.9(ii), $|Z_\alpha \colon Z_\alpha \cap K_4| \geq 4$ and there is $u \in Z_\alpha$ such that u does not induce a transvection either. Considering Z_4 as a symplectic space, we get

$$[Z_4, s] = C_{Z_4}(s)^\perp = (Z_4 \cap K_\alpha)^\perp = C_{Z_4}(u)^\perp = [Z_4, u].$$

If s induces a transvection, clearly $[s, Z_4] = Z_1 \leq [Z_4, Z_\alpha]$. So in any case we have $[Z_4, \tilde{V}_{\alpha+1}] = [Z_4, Z_\alpha] \leq Z_\alpha$ and $\tilde{V}_{\alpha+1} Z_\alpha \trianglelefteq \langle Z_4, G_\alpha \cap G_{\alpha+1} \rangle = G_\alpha$. As above (in Case 1.2) this yields the contradiction $Z_\alpha \leq \tilde{V}_{\alpha+1}$.

It remains to consider

Case 2.2. $\tilde{V}_{\alpha+1} \nleq K_1$; i.e., $\tilde{V}_{\alpha+1}$ acts nontrivially on $\tilde{V}_1/Z_1$. Since $Z_\alpha \leq K_1$, we have $[\tilde{V}_1, Z_\alpha] \leq Z_1$ by (2.1). If $\tilde{V}_1 \nleq K_\alpha$ this means that $\tilde{V}_1$ induces transvections on Z_α. Hence $|\tilde{V}_1 \colon \tilde{V}_1 \cap K_\alpha| \leq 2$.

Further, again by (2.1), $[V_1 \cap K_{\alpha+1}, \tilde{V}_{\alpha+1}] \leq V_1 \cap Z_{\alpha+1} = 1$, so $C_{V_1}(\tilde{V}_{\alpha+1} K_1/K_1) = V_1 \cap K_{\alpha+1}$. Since $|\tilde{V}_1 \colon C_{\tilde{V}_1}(\tilde{V}_{\alpha+1} K_1/K_1)| \geq 4$ by (2.3), there exists an element $t \in \tilde{V}_1 \cap G_{\alpha+1} \setminus K_{\alpha+1}$ and we get

$$|V_{\alpha+1} \colon C_{V_{\alpha+1}}(t)| \leq |V_{\alpha+1} \colon V_{\alpha+1} \cap K_1| \cdot \big|[K_1, t]\big| \leq 2^5 \cdot 2 = 2^6.$$

So $V_{\alpha+1}$, hence also V_1, is an *SC*-module.

Let $W = \langle \tilde{V}_\beta \,|\, \beta \in \Delta(\alpha) \rangle$. Then $W \trianglelefteq G_\alpha$ and $W \leq K_\alpha$. Furthermore, as $b \geq 4$, for $\beta, \gamma \in \Delta(\alpha)$ we have

$$\tilde{V}_\beta = \langle \tilde{Z}_\delta \,|\, \delta \in \Delta(\beta) \rangle \leq \langle \tilde{Z}_\delta \,|\, d(\delta, \gamma) \leq 3 \rangle \leq K_\gamma.$$

Similarly $\tilde{V}_\gamma \leq K_\beta$, so

$$\left[\tilde{V}_\beta, \tilde{V}_\gamma\right] \leq \left[\tilde{V}_\beta, K_\beta\right] \cap \left[\tilde{V}_\gamma, K_\gamma\right] \leq Z_\beta \cap Z_\gamma = 1,$$

which means that W is elementary abelian. This, together with the facts that $V_1 \leq G_\alpha$ and $W \leq G_1$, implies $[V_1, W] \leq V_1 \cap W$ and $[V_1, W, W] = 1$; i.e., W acts quadratically on V_1. So if V_1 involves a natural module we get $|W \colon W \cap K_1| \leq 8$ by Lemma 1.9(ii), if not we even have $|W \colon W \cap K_1| \leq 2$ by (2.4). As $Z_4 \leq G_\alpha$, similarly we see that $W \cap K_1 = W \cap G_4$ acts quadratically on Z_4. Hence $|W \cap K_1 \colon W \cap K_4| \leq 8$ and

$$1 \neq \big|W \colon C_W(Z_4 K_\alpha / K_\alpha)\big| \leq 2^6.$$

In particular, W is an SC-module and N_α acts trivially on W by Lemma 1.6. Now $\tilde{V}_{\alpha+1}Z_\alpha/Z_\alpha$ is a four-dimensional submodule of W/Z_α for $(G_\alpha \cap G_{\alpha+1})/K_\alpha$ and

$$\tilde{V}_{\alpha+1}Z_\alpha/Z_\alpha \cong \tilde{V}_{\alpha+1}/\left(\tilde{V}_{\alpha+1} \cap Z_\alpha\right) = \tilde{V}_{\alpha+1}/\tilde{Z}_\alpha \cong \tilde{Z}_\alpha/Z_{\alpha+1}. \quad (2.11)$$

Moreover, $[\tilde{V}_{\alpha+1}, O_2(G_\alpha \cap G_{\alpha+1}/K_\alpha)] = [\tilde{V}_{\alpha+1}, K_{\alpha+1}K_\alpha/K_\alpha] \leq Z_\alpha$, i.e., $\tilde{V}_{\alpha+1}Z_\alpha/Z_\alpha \leq C_{W/Z_\alpha}(O_2(G_\alpha \cap G_{\alpha+1}/K_\alpha))$. But if M is the natural G_α/N_α-module, then $\dim C_M(O_2(G_\alpha \cap G_{\alpha+1}/K_\alpha)) = 1$, and if M is the spin module, then the action of $G_\alpha \cap G_{\alpha+1}/O_2(G_\alpha \cap G_{\alpha+1})$ on $\tilde{V}_{\alpha+1}Z_\alpha/Z_\alpha$ is not equivalent to the action on $C_M(O_2(\tilde{V}_{\alpha+1}Z_\alpha/Z_\alpha))$. So, by Lemma 1.7, W/Z_α involves $M(\lambda_2)$. As $Z_\alpha \leq W$ is also a nontrivial module, Lemma 1.11 implies that $|Z_4\colon Z_4 \cap K_\alpha| = 2$ and that $s \in Z_4 \setminus K_\alpha$ is of type b_1; i.e., s induces a transvection on Z_α. Now $\tilde{Z}_4 \leq K_\alpha$, and as we may assume that this holds for all $\beta \in \Delta(1)$, we have $\tilde{V}_1 \leq K_\alpha \leq G_{\alpha+1}$.

Furthermore, $\|[W, s]\| \geq 2^5$ by Lemma 1.11. So $K_{14} = K_1K_4$ and $V_1/\tilde{V}_1$ involves the natural module by (2.4). Now $[V_{\alpha+1}, \tilde{V}_1, \tilde{V}_1] \leq [V_{\alpha+1} \cap \tilde{V}_1, \tilde{V}_1] = 1$, i.e., $\tilde{V}_1$ acts quadratically on $V_{\alpha+1}$ and $|\tilde{V}_1\colon \tilde{V}_1 \cap K_{\alpha+1}| \leq 2^3$ by Lemma 1.8(ii). Obviously, this implies that $\tilde{V}_1$ involves only one nontrivial module, hence that $|\tilde{V}_1| = 2^9$.

On the other hand, we have $|\tilde{V}_1\colon \tilde{V}_1 \cap K_{\alpha+1}| = |\tilde{V}_1\colon C_{\tilde{V}_1}(\tilde{V}_{\alpha+1})| \geq 2^2$, so $\|[V_{\alpha+1}, \tilde{V}_1]\| \geq 2^6$ by Lemma 1.10(ii). But $[\tilde{V}_1, V_{\alpha+1}] \leq [\tilde{V}_1, K_0]$ for $0 \in \Delta(1)$ with $d(0, \alpha) = b - 2$ and $\|[\tilde{V}_1, K_0]Z_1/Z_1| = 2^4$, i.e., $\|[\tilde{V}_1, K_0]\| \leq 2^5$. This final contradiction proves the lemma. ∎

LEMMA 2.3. *Let G be a group of type $\tilde{F}_4$. Suppose $C_S(K_i) \leq K_i$ for $i = 1, 4$. Then $\Omega_1(Z(S)) \ntrianglelefteq G_i$ for $i = 1$ and 4.*

Proof. Suppose $\Omega_1(Z(S)) \trianglelefteq G_1$. Then by Lemma 1.5, Z_4 is an F_1-module and, by Lemma 1.6, $[Z_4, N_4] = 1$. Now Lemmas 1.7 and 1.8 and $[\Omega_1(Z(S)), G_{14}] = 1$ yield that Z_4 contains a natural module W_4. Let $T_1 = W_4 \cap Z_1$, $\tilde{W}_4 = [W_4, O_2(G_{14})]$, $\tilde{V}_1 = \langle \tilde{W}_4^{G_1} \rangle$, and $V_1 = \langle W_4^{G_1} \rangle$. Then the statements (2.1)–(2.4) from the proof of Lemma 2.2 hold if we replace Z_1 by T_1 and Z_4 by W_4.

Let $(4, \alpha)$ be a critical pair in $\Gamma(G_1, G_4)$, $b = d(4, \alpha)$, and $d(1, \alpha) = b - 1$. By Lemma 2.2 we have $\alpha \sim 1$.

Suppose $b = 1$, i.e., $Z_4 \nleq K_1$. If $W_4 \leq K_1$, then $[W_4, \tilde{V}_1] \leq [K_1, \tilde{V}_1] \leq T_1$ by (2.1). Thus, $\tilde{V}_1$ can only induce transvections on W_4 and $|\tilde{V}_1\colon \tilde{V}_1 \cap K_4| \leq 2$. But now $1 \neq |\tilde{V}_1\colon C_{\tilde{V}_1}(Z_4K_1/K_1)| \leq 2$ in contradiction to statement (2.3) from Lemma 2.2.

So $W_4 \nleq K_1$. If $\Phi(K_4) \neq 1$, then $\Phi(K_4) \cap \Omega_1(Z(S)) \neq 1$ and we can choose $W_4 \leq \Phi(K_4)$. But $K_4/K_1 \cap K_4 \cong K_4K_1/K_1$ is elementary abelian. Hence $\Phi(K_4) \leq K_1 \cap K_4$. So, as $W_4 \nleq K_1$, we may assume that K_4 is elementary abelian. In particular, $[\tilde{V}_1 \cap K_4, K_4] = 1$ and

$$\left|\tilde{V}_1 : C_{\tilde{V}_1}(K_4K_1/K_1)\right| = |\tilde{V}_1 : \tilde{V}_1 \cap K_4| \leq |K_1K_4/K_1| = |K_4K_1/K_1|;$$

i.e., $\tilde{V}_1$ is an F-module with offending subgroup K_4K_1/K_1. Now $\tilde{V}_1/T_1$ is the spin module by (2.3). As $\tilde{V}_1K_4/K_4 \trianglelefteq G_{14}/K_4$, the structure of $3^7Sp_6(2)$ yields $|\tilde{V}_1 : \tilde{V}_1 \cap K_4| = |\tilde{V}_1K_4/K_4| = 2$ or 2^5. On the other hand, $\tilde{V}_1 \cap K_4 \trianglelefteq G_{14}$ and $T_1 \leq \tilde{V}_1 \cap K_4$. But the spin module does not possess a G_{14}/N_{14}-submodule of index 2 or 2^5, a contradiction.

So $b \geq 3$ and V_1 is elementary abelian. By our choice of α we have $V_1 \leq G_\alpha$ and $V_\alpha \leq G_1$. We are going to reach a contradiction in each of the cases $\tilde{V}_\alpha \leq K_1$ and $\tilde{V}_\alpha \nleq K_1$. Suppose first

Case 1: $\tilde{V}_\alpha \leq K_1 \leq G_4$. Then we have $[\tilde{W}_4, \tilde{V}_\alpha] \leq [\tilde{W}_4, K_1] \leq T_1$ by (2.1). So $\tilde{W}_4 \leq K_\alpha$ by (2.3). We may assume that this holds for any critical pair (β, γ).

Let $W = \langle \tilde{V}_\delta | \delta \in \Delta(4) \rangle = \langle \tilde{W}_\beta | d(4, \beta) = 2 \rangle$. Then $W \trianglelefteq G_4$, $W' = 1$, and $W \leq K_{\alpha-2} \leq G_{\alpha-1}$ since $d(\beta, \alpha - 2) \leq d(\beta, 4) + d(4, \alpha - 2) \leq 2 + (b - 2) = b$ for all such β. In particular, $W \leq G_1$ and $[W, T_1] = 1$.

On the other hand, Z_4 acts nontrivially on V_α. So $|\tilde{V}_\alpha : \tilde{V}_\alpha \cap K_4| = |\tilde{V}_\alpha : C_{\tilde{V}_\alpha}(Z_4K_\alpha/K_\alpha)| \geq 4$ by (2.3) and there is an element $t \in \tilde{V}_\alpha \setminus K_4$ such that t does not induce a transvection on W_4. From the action of $Sp_6(2)$ on the natural module, we see that $T_1 = [\tilde{W}_4, t] \leq [K_\alpha, \tilde{V}_\alpha] \leq T_\alpha$. As $[T_\alpha, G_\alpha] = 1$, we get $T_1 \trianglelefteq \langle W, G_\alpha \rangle$. If $W \nleq G_\alpha$, then $G_{\alpha-1} = \langle W, G_{\alpha-1} \cap G_\alpha \rangle$ and $G = \langle G_{\alpha-1}, G_\alpha \rangle = \langle W, G_\alpha \rangle$. But $T_1 \ntrianglelefteq G$, so we have $W \leq G_\alpha$, $[W, \tilde{V}_\alpha] \leq W \cap \tilde{V}_\alpha \leq C_{\tilde{V}_\alpha}(W)$, and W acts quadratically on $\tilde{V}_\alpha$. Since no elementary abelian subgroup of order 2^6 of $Sp_6(2)$ acts quadratically on the spin module, this yields

$$\left|W : C_W(t)\right| \leq |W : W \cap K_\alpha| \cdot |[K_\alpha, t]| \leq 2^5 \cdot 2 = 2^6.$$

But $\tilde{V}_1W_4/W_4 \cong \tilde{V}_1/\tilde{W}_4$ is a $G_{14}N_1/N_1$-submodule of W/W_4 which is isomorphic to $\tilde{W}_4/T_1$, i.e., to a section of a natural $Sp_6(2)$-module for G_4/N_4. As in the proof of Lemma 2.2, W/W_4 must involve a nontrivial module which can be neither the natural nor the spin module. Hence W involves $M(\lambda_2)$ by Lemma 1.7. As t does not induce a transvection on the natural module, we get $|W/W_4 : C_{W/W_4}(t)| \geq 2^5$ by Lemma 1.11 and $|W : C_W(t)| \geq 2^7$, a contradiction.

It remains to consider.

Case 2: $\tilde{V}_\alpha \nleq K_1$. Let $t \in \tilde{V}_\alpha \setminus K_1$. Then $|\tilde{V}_1 : C_{\tilde{V}_1}(t)| \geq 4$ by (2.3) and $\tilde{V}_1 \nleq K_\alpha$ by (2.1). As $\tilde{V}_1 = \langle \tilde{W}_\beta | \beta \in \Delta(1) \rangle$ we may assume $\tilde{W}_4 \nleq K_\alpha$. On the other hand,

$$\left|\tilde{V}_1\colon C_{\tilde{V}_1}(t)\right| \leq \left|V_1\colon C_{V_1}(t)\right| \leq |V_1\colon V_1 \cap K_\alpha| \cdot \|[K_\alpha, t]\| \leq 2^6$$

and $\tilde{V}_1$ involves the spin module by (2.3).

Let $0 \in \Delta(1)$ such that $d(0, \alpha) = b - 2$. Suppose $|K_1\colon K_1 \cap K_4| = 2$ and choose $s \in Z_4 \setminus K_\alpha$. Then

$$\begin{aligned}\left|\tilde{V}_\alpha\colon C_{\tilde{V}_\alpha}(s)\right| &\leq |\tilde{V}_\alpha\colon \tilde{V}_\alpha \cap K_1| \cdot |\tilde{V}_\alpha \cap K_1\colon \tilde{V}_\alpha \cap K_4| \\ &\leq |K_0\colon K_0 \cap K_1| \cdot |K_1\colon K_1 \cap K_4| \leq 4.\end{aligned}$$

But $\langle s\rangle K_\alpha/K_\alpha = K_{\alpha-1}K_\alpha/K_\alpha = Z(G_{\alpha-1} \cap G_\alpha/K_\alpha)$ and [S1, (1.8)] implies $\|[\tilde{V}_\alpha, s]\| \geq 2^4$, a contradiction.

So $|K_1\colon K_1 \cap K_4| = 2^5$ and by (2.4) $V_1/\tilde{V}_1$ involves the natural module. As $[V_1, V_\alpha] \leq V_1 \cap V_\alpha$ and V_1 and V_α are abelian, V_1 acts quadratically on V_α and vice versa. Hence $|V_1\colon V_1 \cap K_\alpha| \leq 8$ by Lemma 1.9. But the actions of $Sp_6(2)$ on the natural and spin modules yield that $|V_1\colon C_{V_1}(t)| \geq 2^4$ for $t \in \tilde{V}_\alpha \setminus K_1$ as above. Hence

$$|V_1\colon V_1 \cap K_\alpha| = |V_\alpha\colon V_\alpha \cap K_1| = 8, \tag{2.12}$$

$|\tilde{V}_1| = 2^9$, $\tilde{V}_1/T_1$ is the spin module, and $V_1/\tilde{V}_1$ involves, apart from one natural module, only trivial factors.

Let $s \in \tilde{V}_1 \setminus K_\alpha$. Then $|V_\alpha\colon C_{V_\alpha}(s)| = 2^4$ and $|\tilde{V}_\alpha\colon C_{\tilde{V}_\alpha}(s)| \leq 2^3$. Therefore, $\tilde{V}_1 N_\alpha/N_\alpha$ induces only involutions of type a_2 on $\tilde{V}_\alpha$ and $|\tilde{V}_1\colon \tilde{V}_1 \cap K_\alpha| \leq 4$ by Lemma 1.9.

Suppose $|\tilde{V}_1\colon \tilde{V}_1 \cap K_\alpha| = 4$. Then $\|[V_\alpha, \tilde{V}_1]\| \geq 2^6$ by Lemma 1.10. But $[V_\alpha, \tilde{V}_1] \leq [K_0, \tilde{V}_1] = \tilde{V}_1 \cap W_0$ and $|\tilde{V}_1 \cap W_0| = 2^5$, a contradiction. As the same holds interchanging the roles of 1 and α and substituting 0 by $\alpha - 1$, we have

$$|\tilde{V}_1\colon \tilde{V}_1 \cap K_\alpha| = |\tilde{V}_\alpha\colon \tilde{V}_\alpha \cap K_1| = 2. \tag{2.13}$$

Now $[\tilde{V}_1 \cap K_\alpha, \tilde{V}_\alpha] \neq 1 \neq [\tilde{V}_1, \tilde{V}_\alpha \cap K_1]$ and $T_1 T_\alpha \leq \tilde{V}_1 \cap \tilde{V}_\alpha$.

Let $t \in V_\alpha \setminus K_1$ of type c_2 (t exists by (2.12) and Lemma 1.9). Then $\|[\tilde{V}_1/T_1, t]\| = 2^4$, and from $T_1 \leq [\tilde{V}_1, \tilde{V}_\alpha]$ it follows that $[\tilde{V}_1, V_\alpha] = \tilde{V}_1 \cap W_0$. But from (2.13) we deduce $\|[\tilde{V}_1 \cap K_\alpha, t]T_1/T_1| \geq 2^3$. As $[\tilde{V}_1 \cap K_\alpha, t]T_1 \leq \tilde{V}_\alpha$ this yields

$$\left|\left[\tilde{V}_1, V_\alpha/\tilde{V}_\alpha\right]\right| = \left|\left[\tilde{V}_1, V_\alpha\right]\tilde{V}_\alpha/\tilde{V}_\alpha\right| = \left|\left[\tilde{V}_1, V_\alpha\right]\colon \left[\tilde{V}_1, V_\alpha\right] \cap \tilde{V}_\alpha\right| = 2;$$

i.e., $\tilde{V}_1 N_\alpha/N_\alpha$ induces a transvection on $V_\alpha/\tilde{V}_\alpha$. This contradicts the previously mentioned fact that $\tilde{V}_1$ can only induce involutions of type a_2 and the lemma is proved. ∎

LEMMA 2.4. *Let G be a group of type $\tilde{F}_4$. Suppose $C_S(K_i) \leq K_i$ and $Z_i \neq \Omega_1(Z(S))$ for $i = 1, 4$. Then $Z_1 \nleq K_4$, $Z_4 \nleq K_1$, and $Z_i/Z(G_i)$ are natural $Sp_6(2)$-modules for $i = 1, 4$.*

Proof. The assumption together with Lemma 1.5 implies that Z_1 and Z_4 are F_1-modules. Hence, by Lemma 1.6, $[Z_1, N_1] = [Z_4, N_4] = 1$ and by Lemma 1.17, Z_1 and Z_4 involve only natural and spin modules as nontrivial factors.

Look at the coset graph $\Gamma = \Gamma(G_1, G_4)$. By symmetry, we may assume that there exists a critical pair $(1, \alpha)$.

Suppose Z_1 involves the spin module. Then Lemma 1.7 yields that Z_1 involves only one nontrivial G_1/N_1-factor; i.e., $Z_1/Z(G_1)$ is the spin module. Now $[\Omega_1(Z(S)), \langle P_3, P_4\rangle] = 1$ by Lemma 1.8. Further, by Lemma 1.3, $[Z_1, Z_\alpha] \neq 1$ and, by [S1, (1.9)], Z_α cannot involve the spin module. So $\alpha \sim 4$ and Z_4 involves only natural and trivial modules. As there are no transvections on the spin module, we have $|Z_1\colon Z_1 \cap K_\alpha| \geq 4$. Now Lemma 1.9(ii) yields $|Z_\alpha\colon Z_\alpha \cap K_1| \geq 4$. Hence $|Z_1\colon Z_1 \cap K_\alpha| \geq 2^3$ by the action of $Sp_6(2)$ on the spin module and $|Z_\alpha\colon Z_\alpha \cap K_1| \geq 2^3$ again by Lemma 1.9(ii). Now $Z_\alpha N_1/N_1$ contains an element of type c_2 and we even get $|Z_1\colon Z_1 \cap K_\alpha| \geq 2^4$. Since $|Z_\alpha\colon Z_\alpha \cap K_1| \leq 2^5$, this implies that Z_α (and also Z_4) involves only one nontrivial factor; i.e., $Z_4/Z(G_4)$ is the natural module. From Lemma 1.8 we get $[\Omega_1(Z(S)), \langle P_2, P_3\rangle] = 1$ and $\Omega(Z(S)) \trianglelefteq \langle P_2, P_3, P_4\rangle = G_1$, a contradiction.

Therefore Z_1 involves only natural and trivial modules. By symmetry, the same holds for Z_α. From Lemma 1.9(ii) we deduce that

$$|Z_1\colon Z_1 \cap K_\alpha| = |Z_\alpha\colon Z_\alpha \cap K_1| \tag{2.14}$$

and that $Z_1/Z(G_1)$ and $Z_\alpha/Z(G_\alpha)$ are natural modules. By Lemma 1.8 we get $[\Omega_1(Z(S)), G_{14}] = 1$. So, again by Lemmas 1.7 and 1.8, Z_4 involves as nontrivial factors at most two natural modules. Further $Z_1 \cap Z_4 \trianglelefteq G_{14}$, but the action of $G_{14}/N_{14} \cong \Sigma_6$ on the natural module for G_1/N_1 is not equivalent to the action on the natural module for G_4/N_4. So $Z_1 \cap Z_4 = \Omega_1(Z(S))$.

Let $b = d(1, \alpha)$. We want to show that $b = 1$. Then we may take $\alpha = 4$ and the assertion of the lemma follows from what we have already proved about the structures of Z_1 and Z_α. We are going to exclude the cases $b = 2$ and $b \geq 3$ separately. Suppose first

Case 1: $b = 2$, i.e., $\alpha \in \Delta(4)$. Let $W_4 \leq Z_4$ such that $W_4 \trianglelefteq G_4$ and $W_4/(W_4 \cap Z(G_4))$ is the natural module. Let $\tilde{W}_4 = [W_4, K_{14}]$ and $V_1 = \langle \tilde{W}_4^{G_1}\rangle$. Then $V_1 \leq K_1$ because of $b > 1$. Further

$$[V_1, K_1] \leq \left[\tilde{W}_4, K_1\right]^{G_1} \leq \langle \Omega_1(Z(S))^{G_1}\rangle = Z_1,$$

which means that $K_1 \leq C_{G_1}(V_1/Z_1)$. Since G_{14}/N_{14} acts nontrivially on $\tilde{W}_4/(\tilde{W}_4 \cap \Omega_1(Z(S))) = \tilde{W}_4/(\tilde{W}_4 \cap Z_1)$, we get $K_1 = C_S(V_1/Z_1)$. In particular, V_1/Z_1 involves a nontrivial G_1/K_1-module. As in the former lemmas the action of $G_{14}N_1/N_1$ on $\tilde{W}_4/(\tilde{W}_4 \cap Z(G_4))$ shows that this module cannot be the natural $Sp_6(2)$-module.

From $Z_4 \leq K_1$ and $V_1 \leq G_4$ we get

$$[V_1, Z_4] \leq [V_1, K_1] \cap [G_4, Z_4] \leq Z_1 \cap Z_4 = \Omega(Z(S)).$$

If $V_1 \not\leq K_4$ this implies that V_1 induces transvections on W_4. Thus $|V_1\colon V_1 \cap K_4| = 2$, $V_1N_4/N_4 = Z(G_{14}N_4/N_4)$, and $[V_1, \tilde{W}_4] = 1$. Now also $V_1' = 1$.

As $V_1 \cap K_4 \leq G_\alpha$ and $Z_\alpha \leq G_1$, we get

$$[Z_\alpha, V_1 \cap K_4, V_1 \cap K_4] \leq [Z_\alpha \cap V_1, V_1 \cap K_4] \leq V_1' = 1.$$

So $V_1 \cap G_\alpha$ acts quadratically on Z_α and Lemma 1.9 yields

$$\begin{aligned} &|V_1 \cap G_\alpha : V_1 \cap K_\alpha| \leq 2^3, \\ &\left|V_1\colon C_{V_1}(Z_\alpha)\right| = |V_1\colon V_1 \cap K_4| \cdot |V_1 \cap G_\alpha : V_1 \cap K_\alpha| \leq 2^4. \end{aligned} \tag{2.15}$$

As $[Z_1, Z_\alpha] \neq 1$ we get $|V_1/Z_1\colon C_{V_1/Z_1}(Z_\alpha)| \leq 2^3$; in particular, V_1/Z_1 is an SC-module which by Lemma 1.11 cannot involve $M(\lambda_2)$. But as we remarked previously, it must involve a module which is not the natural one. So V_1/Z_1 involves the spin module. Now Lemma 1.10(ii) implies $|Z_\alpha\colon Z_\alpha \cap K_1| = 2$ since otherwise we would get a contradiction to (2.15). Hence, by (2.14), $Z_\alpha N_1/N_1$ induces a transvection on Z_1. But now [S1, (1.8)] yields $|V_1/Z_1\colon C_{V_1/Z_1}(Z_\alpha N_1/N_a)| \geq 2^4$ and we get $|V_1\colon C_{V_1}(Z_\alpha)| \geq 2^5$, also a contradiction to (2.15).

Case 2: $b \geq 3$. For $\{i, j\} = \{1, 4\}$ let $V_i = \langle Z_j^{G_i} \rangle$. Then $V_i' = 1$ because of $b \geq 3$. As in Case 1, V_i/Z_i involves a nontrivial G_i/K_i-module which is not the natural $Sp_6(2)$-module. Moreover, $V_1 \leq G_{\alpha-1}$ and $V_{\alpha-1} \leq G_1$. Without loss of generality we may assume that $d(4, \alpha) = b - 1$.

Suppose there does not exist any critical pair of type $(4, \beta)$. Then $(\delta, \alpha - 1)$ is not a critical pair for each $\delta \in \Delta(1)$, i.e., $Z_\delta \leq K_{\alpha-1} \leq G_\alpha$. Hence also $V_1 = \langle Z_\delta | \delta \in \Delta(1) \rangle \leq G_\alpha$. As $Z_\alpha \leq G_1$ and V_1 is abelian, we get $[Z_\alpha, V_1, V_1] \leq [Z_\alpha \cap V_1, V_1] = 1$; i.e., V_1 acts quadratically on Z_α. Now Lemma 1.9 implies $2^3 \geq |V_1\colon V_1 \cap K_\alpha| = |V_1\colon C_{V_1}(Z_\alpha K_1/K_1)|$. Since $Z_\alpha \not\leq K_1$, this yields a contradiction as at the end of Case 1.

So there is also a critical pair $(4, \beta)$ and all the things said about Z_1 hold for Z_4, too. In particular, $Z_4/Z(G_4)$ is the natural module and $|\Omega(Z(S))| \leq 4$.

For $i = 1, 4$. Let $\tilde{Z}_i \leq Z_i$ with $|Z_i : \tilde{Z}_i| = 2$ and $\tilde{Z}_i \trianglelefteq G_{14}$. Set $\tilde{V}_i = \langle \tilde{Z}_j^{G_i} \rangle$ for $\{i, j\} = \{1, 4\}$. As in Case 1 we can show

$$\left[\tilde{V}_i, K_i\right] \leq Z_i \qquad \text{and} \qquad 2^5 \leq |\tilde{V}_\alpha : \tilde{V}_\alpha \cap K_1| \tag{2.16}$$

(because $\tilde{V}_i$ contains apart from Z_i a nontrivial module which is not the natural one). In particular, $K_{14} = K_1 K_4$. As $Z_1 Z_4 \leq G_\alpha$, $V_\alpha \leq G_4$, and V_α is abelian, V_α acts quadratically on Z_4 and $V_\alpha \cap G_1$ acts quadratically on Z_1. Applying Lemma 1.9(ii) twice, we get

$$\begin{aligned} |V_\alpha : V_\alpha \cap K_1| &= |V_\alpha : V_\alpha \cap K_4| \cdot |V_\alpha \cap K_4 : V_\alpha \cap K_1| \\ &\leq 2^3 \cdot 2^3 = 2^6. \end{aligned} \tag{2.17}$$

Combining this with (2.16), one easily sees that $V_\alpha \not\leq G_1$.

Suppose $|Z_\alpha : Z_\alpha \cap K_1| \geq 4$. Then G_1/K_1 acts trivially on $V_1/\tilde{V}_1$, because otherwise, interchanging 1 and α in (2.16), we would get

$$\begin{aligned} |V_1 : V_1 \cap K_\alpha| &\geq \left| V_1/\tilde{V}_1 : C_{V_1/\tilde{V}_1}(Z_\alpha K_1/K_a) \right| \cdot |\tilde{V}_1 : \tilde{V}_1 \cap K_\alpha| \\ &\geq 2^2 \cdot 2^5 = 2^7, \end{aligned}$$

in contradiction to (2.17). So $\tilde{V}_1 \langle t \rangle \trianglelefteq G_1$ where $t \in Z_4 \setminus \tilde{Z}_4$. Now also $[\tilde{V}_1 \langle t \rangle, K_1] \trianglelefteq G_1$. But we have $[\tilde{V}_1 \langle t \rangle, K_1] = [\tilde{V}_1, K_1][t, K_1] \leq Z_1 \tilde{Z}_4$ and $\tilde{Z}_4 = [t, K_1] Z(G_4) \not\leq Z_1$, also a contradiction.

Hence $|Z_\alpha : Z_\alpha \cap K_1| = |Z_1 : Z_1 \cap K_\alpha| = 2$ and $\tilde{Z}_1 \leq K_\alpha$. From (2.16) and (2.17) we deduce that $|\tilde{V}_\alpha \cap G_1 : \tilde{V}_\alpha \cap K_1| = |\tilde{V}_\alpha \cap K_4 : \tilde{V}_\alpha \cap K_1| \geq 4$. So the action of $Sp_6(2)$ on the natural module yields

$$1 \neq \left[\tilde{V}_\alpha \cap K_4, \tilde{Z}_1\right] \leq \left[K_4, \tilde{V}_4\right] \cap [G_1, Z_1] \leq Z_1 \cap Z_4 = \Omega_1(Z(S)).$$

On the other hand, $[\tilde{V}_\alpha, \tilde{Z}_1] \leq V_\alpha$ and V_α is abelian. So, as $\Omega_1(Z(S)) \trianglelefteq G_{14}$ and $V_\alpha \subseteq G_4 \setminus G_{14}$, we get $[\tilde{V}_\alpha \cap K_4, \tilde{Z}_1] \trianglelefteq \langle V_\alpha, G_{14} \rangle = G_4$ and $Z(G_4) \neq 1$. But $\tilde{Z}_1 \leq K_\alpha$; therefore

$$Z(G_4) = \left[\tilde{V}_\alpha \cap K_4, \tilde{Z}_1\right] \leq \left[\tilde{V}_\alpha, K_\alpha\right] \leq Z_\alpha.$$

Let $\alpha + 1 \in \Delta(\alpha)$ and $\alpha \neq \beta \in \Delta(\alpha + 1)$. Then $d(\beta, \alpha) = 2 < b$ and $[Z(G_4), Z_\beta] \leq [Z_\alpha, Z_\beta] = 1$. Choose $0 \in \Delta(4)$ such that $d(0, \alpha) = b - 2$. Then $d(\beta, 0) \leq d(\beta, \alpha) + d(\alpha, 0) = 2 + (b - 2) = b$ and $Z_\beta \leq G_0$. So $Z_\beta \leq C_{G_0}(Z(G_4)) = G_0 \cap G_4$. Therefore $V_{\alpha+1} = \langle Z_\beta \mid \beta \in \Delta(\alpha + 1) \rangle \leq G_4$. As we have seen, $V_\delta \not\leq G_\gamma$ for any critical pair (δ, γ) (compare the remark after (2.17)), this implies that the pair $(\alpha + 1, 4)$ is not a critical pair, i.e., $Z_{\alpha+1} \leq K_4 \leq G_1$.

As $Z_4 \le G_{\alpha+1}$, we get $[V_{\alpha+1}, Z_4] \le Z_4 \cap V_{\alpha+1}$ and $V_{\alpha+1}$ acts quadratically on Z_4. Now Lemma 1.9 yields $|V_{\alpha+1}\colon V_{\alpha+1} \cap K_4| \le 2^3$. Further,

$$\left|[V_{\alpha+1} \cap K_4, Z_1 \cap G_{\alpha+1}]\right| = \left|\left[V_{\alpha+1} \cap K_, \tilde{Z}_1\right]\right| \le 2.$$

So we get $|V_{\alpha+1}\colon C_{V_{\alpha+1}}(t)| \le 2^4$ for $t \in Z_1 \cap G_{\alpha+1} = Z_1 \cap K_\alpha$. If $\tilde{Z}_1 = Z_1 \cap K_\alpha \not\le K_{\alpha+1}$ this implies that $G_{\alpha+1}$ acts trivially on $V_{\alpha+1}/\tilde{V}_{\alpha+1}$. But, as we have seen previously, this yields a contradiction, and therefore $\tilde{Z}_1 \le K_{\alpha+1}$. Now $Z_{\alpha+1}$ induces transvections on Z_1. Hence $[Z_{\alpha+1}, Z_1] = [Z_\alpha, Z_1] \le Z_\alpha$ and $Z_\alpha Z_{\alpha+1} \trianglelefteq \langle G_\alpha \cap G_{\alpha+1}, Z_1\rangle = G_\alpha$. But now also $[Z_\alpha Z_{\alpha+1}, K_\alpha] \trianglelefteq G_\alpha$. Because of $[Z_\alpha Z_{\alpha+1}, K_\alpha] = [Z_{\alpha+1}, K_\alpha] = \tilde{Z}_{\alpha+1}$, this yields a contradiction and the lemma is proved. ∎

LEMMA 2.5. *Let G be a group of type $\tilde{F}_4$. Then $C_S(K_i) \le K_i$ for $i = 1, 4$.*

Proof. Suppose the assertion is false. By symmetry, we assume that $C_S(K_1) \not\le K_1$. Then $G_1 = C_{G_1}(K_1)$, $[O^2(G_{14}), K_1] = 1$, and from the action of G_{14}/K_{14} on K_{14}/K_4 we get

$$|K_1\colon K_1 \cap K_4| = |K_4\colon K_4 \cap K_1| \le 2.$$

Hence $[K_4, G_{14}] \le K_1 \cap K_4$ and $[K_4, O^2(G_{14})] = 1$. In particular, we have $C_S(K_4) \not\le K_4$, too. Now $G_i = C_{G_i}(K_i)S$ holds for $i = 1, 4$, and we get $K_1 \cap K_4 \trianglelefteq \langle O^2(G_1), O^2(G_4), S\rangle = G$. So $K_1 \cap K_4 = 1$, $|K_1| = |K_4| \le 2$, and $|S| = 2^9$ or 2^{10}.

Now we look at G_2. Since $G_2/K_2 \cong \Sigma_3 \times L_3(2)$, we have $|S\colon K_2| = 2^4$ and $|K_2| = 2^5$ resp. 2^6. But $K_2K_1 \trianglelefteq G_{12}$, so from the action of G_{12}/K_{12} on K_{12}/K_1 it follows that $|K_2K_1/K_1| \in \{1, 2^3, 2^6\}$. One easily sees that the only possibility is $|K_1| = 2$, $|K_2| = 2^6$, $K_1 \not\le K_2$. Replacing 2 by 3 and 1 by 4, analogously we get $|K_4| = 2$, $|K_3| = 2^6$, $K_4 \not\le K_3$. Then K_1K_2 and K_3K_4 are elementary abelian subgroups of S of order 2^7. Since by [Gil, (2.5)] the 2-rank of $Sp_6(2)$ is 6 and every Sylow-2-subgroup of $Sp_6(2)$ contains exactly one elementary abelian subgroup of order 2^6, we get the contradiction $K_1K_2 = K_3K_4 \trianglelefteq \langle G_{12}, G_{34}\rangle = G$. ∎

LEMMA 2.6. *Let G be a group of type $\tilde{F}_4$. Then $K_{14} = K_1K_4$ and for $i = 1$ and 4 the following hold*:

(i) $N_i = O_3(G_i) \times K_i$.

(ii) $|Z(G_i)| = 2$, $|Z(K_i)| = 2^7$, *and* $|K_i| = 2^{15}$.

(iii) $K_i' = \Phi(K_i) = Z(G_i)$.

(iv) *$Z(K_i)/Z(G_i)$ is the natural $Sp_6(2)$-module, $K_i/Z(K_i)$ is the spin module, and G_i/N_i acts indecomposably on $Z(K_i)$ and on $K_i/Z(G_i)$.*

Proof. By Lemma 2.5 we have $C_S(K_i) \leq K_i$ and, by Lemma 2.3, $Z_i \neq \Omega_1(Z(S))$ for $i = 1, 4$. Now, by Lemma 2.4, $Z_i \not\leq K_j$ and $Z_i/Z(G_i)$ is the natural $Sp_6(2)$-module for $\{i, j\} = \{1, 4\}$. As $Z_i K_j/K_j \trianglelefteq G_{14}/K_j$ and $Z_i K_j/K_j \cong Z_i/Z_i \cap K_j$ as G_{14}/N_{14}-module, we get $|Z_i\colon Z_i \cap K_j| = 2$ from the structure of G_{14}/K_i and the action of $Sp_6(2)$ on the natural module. To be more precise, $(Z_i \cap K_j)Z(G_i)/Z(G_i) = [Z_i, K_{14}]Z(G_i)/Z(G_i)$ and $[Z_i \cap K_j, K_{14}] \leq \Omega_1(Z(S))$. Let $t_i \in Z_i \setminus K_j$. Then

$$1 \neq \left|K_i\colon C_{K_i}(t_j)\right| \leq |K_i\colon K_i \cap K_j| \leq 2^5 \tag{2.18}$$

and $O_3((G_i/K_i)$ acts trivially on K_i by Lemma 1.6, i.e., $N_i = O_3(G_i) \times K_i$, and (i) is shown.

Let $V_i = \langle (Z_j \cap K_i)^{G_i} \rangle$. Then

$$V_i' \leq [V_i, K_i] \leq \left[Z_j \cap K_i, K_i\right]^{G_i} \leq \langle \Omega_1(Z(S))^{G_i} \rangle = Z_i.$$

Therefore we can regard V_i/Z_i as a module for G_i/N_i. As G_{14} acts nontrivially on $Z_j \cap K_i/Z_1 \cap Z_4$, V_i/Z_i is a nontrivial module and $|V_i\colon V_i \cap K_j| > |Z_i\colon Z_i \cap K_j| = 2$. Hence $K_{14} = V_1 K_4 = V_4 K_1$ by the action of G_{14}/K_i on K_{14}/K_i.

Now $[V_i, t_j] = [K_{14}, t_j] = Z_j \cap K_i$ and $[V_i, t_j]Z_i/Z_i \not\cong [M_i, t_j]$, where M_i is a G_i/N_i-module with $M_i \cong M(\lambda_2)$. So (2.18) implies that V_i/Z_i involves a spin module and that K_i does not involve any other nontrivial modules. By Lemma 1.12 we may assume that there are $Z_i \leq W_i \trianglelefteq G_i$ such that K_i/W_i is the spin module and W_i/Z_i is a trivial module, i.e., $[G_i, W_i] \leq Z_i$.

Taking into account that $K_1/K_1 \cap K_4 \cong K_1 K_4/K_4$ is elementary abelian, we deduce from the action of K_{14}/K_4 on Z_4 that

$$1 \neq [K_1, Z_4 \cap K_1] \leq K_1' \leq \Phi(K_1) \leq K_1 \cap K_4.$$

Now $\Phi(K_1) \cap \Omega_1(Z(S)) \neq 1$, and since $Z_1 \not\leq K_4$, we get $\Phi(K_1) \leq Z(G_1)$. But $|Z(G_1)| \leq 2$, and hence $Z(G_1) = \Phi(K_1) = K_1'$ is a group of order 2. By symmetry, the same holds for G_4 and (iii) is shown.

Next we show that $W_i = Z_i$. We have $[Z_4, W_1] \leq Z_4 \cap [G_{14}, W_1] \leq Z_4 \cap Z_1 = \Omega_1(Z(S))$; i.e., W_1 induces transvections on $Z_4/Z(G_4)$. So we get $|W_1\colon W_1 \cap K_4| = 2$ and $W_1 = Z_1(W_1 \cap K_4)$. Now $\Phi(W_1) \leq \Phi(W_1 \cap K_4) \leq \Phi(K_4) = Z(G_4)$ and $\Phi(W_1) = 1$. Moreover,

$$[Z_4 \cap K_1, W_1] = \left[Z_4 \cap K_1, (W_1 \cap K_4)Z_1\right] = [Z_4 \cap K_1, Z_1] = 1.$$

Hence

$$[K_1, W_1] = [V_1 W_1, W_1] = [V_1, W_1] \leq [Z_4 \cap K_1, W_1]^{G_1} = 1.$$

In particular, since K_1/W_1 is an irreducible G_1-module and $K_1' \neq 1$, we have $W_1 = Z(K_1)$. As $|W_1\colon C_{W_1}(t_4)| = |W_1\colon W_1 \cap K_4| = 2$ and $G_1/N_1 \cong Sp_6(2)$ can be generated by seven conjugates of t_4, we get $|W_1\colon W_1 \cap Z(G_1)| \leq 2^7$ and $|W_1| \leq 2^8$. Suppose $|W_1| = 2^8$. Then $|K_1| = |K_4| = 2^{16}$, $|K_1 \cap K_4| = 2^{11}$, and $|W_4| = 2^8$, too. This yields

$$|W_1 \cap W_4| = \frac{|W_1 \cap K_4| \cdot |W_4 \cap K_1|}{|(W_1 \cap K_4)(W_4 \cap K_1)|} \geq \frac{2^7 \cdot 2^7}{|K_1 \cap K_4|} = 2^3,$$

in contradiction to $|\Omega_1(Z(S))| = 4$. Thus $|W_i| = 2^7$ and we have (ii).

To complete the proof, it remains to show that G_i/N_i acts indecomposably on Z_i and $K_i/Z(G_i)$. Let $Z(G_i) = \langle z_i \rangle$. Suppose $Z_1 = \langle z_1 \rangle \times U_1$ with $U_1 = [Z_1, G_1]$. We can choose $t_1 \in U_1 \setminus K_4$. Let $t_4 \in Z_4 \setminus K_1$ as above. Then $1 \neq [t_1, t_4] \in C_{U_1}(S/K_1)$ and $1 \neq [t_1, t_4]\langle z_4 \rangle / \langle z_4 \rangle \in C_{Z_4/\langle z_4 \rangle}(S/K_4)$. As $\Omega_1(Z(S)) = \langle z_1, z_4 \rangle$ we must have $[t_1, t_4] = z_1 z_4$. Now the action of K_{14}/K_1 on U_1 yields $z_1 z_4 = [U_1 \cap K_4, K_{14}] = [U_1 \cap K_4, K_4] \leq K_4' = Z(G_4) = \langle z_4 \rangle$, a contradiction. So G_1 acts indecomposably on Z_1 and, by symmetry, the same holds for the action of G_4 on Z_4.

Now suppose $K_1 = Z_1 U_1$ with $Z_1 \cap U_1 = \langle z_1 \rangle$ and $U_1 = [U_1, G_1]$. Then the action of $Sp_6(2)$ on the spin module implies that

$$[U_1, K_{14}]\langle z_1 \rangle / \langle z_1 \rangle = [U_1, t_4]\langle z_1 \rangle / \langle z_1 \rangle.$$

As $[U_1, t_4] \leq Z_4$, this means that U_1 acts trivially on K_4/Z_4, in contradiction to $U_1 \nleq K_4$. Again by symmetry, the second statement of (iv) follows. ∎

3. UNIQUENESS OF THE AMALGAM $\{G_1, G_2, G_3, G_4\}$

Now we are going to determine the multiplication tables of G_1 and G_4. Here we partly follow the construction in [Wes]. For $i = 1, 4$ let $\langle z_i \rangle = Z(G_i)$, $Z_i = Z(K_i)$, and $M_i = O_3(G_i)$. By Lemma 2.6 we have $|M_i| = 3^7$, $|Z_i| = 2^7$, and $[M_i, K_i] = 1$. Moreover, $K_i/Z_i \cong K_i M_i / Z_i M_i = N_i / Z_i M_i$ is the spin module for $G_i/N_i \cong Sp_6(2)$ and [Wes, (1.5)] yields that $G_i/Z_i M_i$ splits over $N_i/Z_i M_i$. As the extension $3^7 Sp_6(2)$ is nonsplit, we get $G_i = K_i \tilde{G}_i$ with $\tilde{G}_i \cap K_i \leq Z_i$. Our first step will be to show that G_i splits over K_i.

First of all, for $i = 2, 3, 4$ we have $P_i = P_i \cap G_1 = P_i \cap K_1 \tilde{G}_1 = K_1(P_i \cap \tilde{G}_1) = K_1 U_i$, where $U_i = P_i \cap \tilde{G}_1$. Similarly, $S = K_1 S_1$ with $S_1 = S \cap \tilde{G}_1$. Let $\overline{G}_1 = \tilde{G}_1 / M_1 Z_1$ and denote by $\overline{H}, \overline{h}$ the images in $\overline{G}_1$ of any subgroup $H \leq \tilde{G}_1$ resp. any element $h \in \tilde{G}_1$. Then $\{\overline{U}_2, \overline{U}_3, \overline{U}_4\}$ are the minimal

TABLE I

$\bar{t}_i$	$\bar{t}_i^{\bar{u}_2}$	$\bar{t}_i^{\bar{u}_3}$	$\bar{t}_i^{\bar{u}_4}$
$\bar{t}_1$	$\bar{t}_1$	$\bar{t}_1$	$\bar{t}_6$
$\bar{t}_2$	$\bar{t}_2$	$\bar{t}_3$	$\bar{t}_2$
$\bar{t}_3$	$\bar{t}_4$	$\bar{t}_2$	$\bar{t}_7$
$\bar{t}_4$	$\bar{t}_3$	$\bar{t}_5$	$\bar{t}_9$
$\bar{t}_5$	$\bar{t}_5$	$\bar{t}_4$	–
$\bar{t}_6$	$\bar{t}_6$	$\bar{t}_8$	$\bar{t}_1$
$\bar{t}_7$	$\bar{t}_9$	$\bar{t}_7$	$\bar{t}_3$
$\bar{t}_8$	–	$\bar{t}_6$	$\bar{t}_8$
$\bar{t}_9$	$\bar{t}_7$	–	$\bar{t}_4$

	x_1	x_2	x_3	x_4	x_5	x_6
t_1	1	1	1	1	1	x_1z_1
t_2	1	1	1	1	x_1	x_2
t_3	1	1	1	x_1	1	x_3
t_4	1	1	x_1	1	1	x_4
t_5	1	x_1	1	1	1	x_5
t_6	1	1	1	1	x_2z_1	1
t_7	1	1	1	x_2	x_3	1
t_8	1	1	1	x_3z_1	1	1
t_9	1	1	x_2	1	x_4	1
u_2	1	1	x_3x_4	x_3x_4	1	1
u_3	1	x_2x_3	x_2x_3	x_4x_5	x_4x_5	1
u_4	x_1x_2	x_1x_2	1	1	x_5x_6	x_5x_6

parabolics of $\overline{G}_1 \cong Sp_6(2)$ with common Sylow-2-subgroup $\overline{S}_1$. We can choose elements $t_1, \ldots, t_9 \in S_1$ and $u_i \in U_i$, $i = 2, 3, 4$, such that $\overline{G}_1 = \langle \bar{t}_1, \ldots, \bar{t}_9, \bar{u}_2, \bar{u}_3, \bar{u}_4 \rangle$ and the following relations hold (cf. [Wes]):

$$1 = (\bar{u}_2\bar{u}_4)^2 = (\bar{u}_3\bar{u}_4)^3 = (\bar{u}_2\bar{u}_3)^4, \tag{3.1}$$

$$\begin{aligned} &[\bar{t}_3, \bar{t}_9] = \bar{t}_2, \qquad [\bar{t}_4, \bar{t}_8] = \bar{t}_1\bar{t}_3, \qquad [\bar{t}_4, \bar{t}_7] = \bar{t}_2, \\ &[\bar{t}_4, \bar{t}_6] = \bar{t}_1\bar{t}_2, \qquad [\bar{t}_5, \bar{t}_7] = \bar{t}_3, \qquad [\bar{t}_5, \bar{t}_9] = \bar{t}_4, \\ &[\bar{t}_8, \bar{t}_9] = \bar{t}_6\bar{t}_7. \end{aligned} \tag{3.2}$$

All the other commutators between the $\bar{t}_i$ are trivial and $\bar{u}_2, \bar{u}_3, \bar{u}_4$ act as indicated on the left in Table I on $\{\bar{t}_1, \ldots, \bar{t}_9\}$ (where a "–" means that $o(\bar{t}_i\bar{u}_j) = 3$ for the respective elements).

Since, by Lemma 2.6(iv), Z_1 is an indecomposable $\overline{G}_1$-module, by [Dem] the action of $\overline{G}_1$ on Z_1 is uniquely determined and we can choose $x_1, \ldots, x_6 \in Z_1$ such that the relations on the right of Table I hold (compare [Wes, Table IV]). Notice that $[Z_1, N_1] = 1$ by Lemma 2.6, so these relations are even independent of the choice of $g \in gN_1$ for $g \in \tilde{G}_1$.

Now we do the same for G_4. Let $\overline{G}_4 = \tilde{G}_4/M_4Z_4$, $V_i = P_i \cap \tilde{G}_4$ for $i = 1, 2, 3$, $S_4 = S \cap \tilde{G}_4$, and choose $s_1, \ldots, s_9 \in S_4$, $v_i \in V_i$, $y_1, \ldots, y_6 \in Z_4$ such that the relations in Table II hold and

$$1 = (\bar{v}_1\bar{v}_3)^2 = (\bar{v}_1\bar{v}_2)^3 = (\bar{v}_2\bar{v}_3)^4, \tag{3.3}$$

TABLE II

$\bar{s}_i$	$\bar{s}_i^{\bar{v}_1}$	$\bar{s}_i^{\bar{v}_2}$	$\bar{s}_i^{\bar{v}_3}$
$\bar{s}_1$	$\bar{s}_6$	$\bar{s}_1$	$\bar{s}_1$
$\bar{s}_2$	$\bar{s}_2$	$\bar{s}_3$	$\bar{s}_2$
$\bar{s}_3$	$\bar{s}_7$	$\bar{s}_2$	$\bar{s}_4$
$\bar{s}_4$	$\bar{s}_9$	$\bar{s}_5$	$\bar{s}_3$
$\bar{s}_5$	–	$\bar{s}_4$	$\bar{s}_5$
$\bar{s}_6$	$\bar{s}_1$	$\bar{s}_8$	$\bar{s}_6$
$\bar{s}_7$	$\bar{s}_3$	$\bar{s}_7$	$\bar{s}_9$
$\bar{s}_8$	$\bar{s}_8$	$\bar{s}_6$	–
$\bar{s}_9$	$\bar{s}_4$	–	$\bar{s}_7$

	y_1	y_2	y_3	y_4	y_5	y_6
s_1	1	1	1	1	1	$y_1 z_4$
s_2	1	1	1	1	y_1	y_2
s_3	1	1	1	y_1	1	y_3
s_4	1	1	y_1	1	1	y_4
s_5	1	y_1	1	1	1	y_5
s_6	1	1	1	1	$y_2 z_4$	1
s_7	1	1	1	y_2	y_3	1
s_8	1	1	1	$y_3 z_4$	1	1
s_9	1	1	y_2	1	y_4	1
v_1	$y_1 y_2$	$y_1 y_2$	1	1	$y_5 y_6$	$y_5 y_6$
v_2	1	$y_2 y_3$	$y_2 y_3$	$y_4 y_5$	$y_4 y_5$	1
v_3	1	1	$y_3 y_4$	$y_3 y_4$	1	1

$$\begin{aligned} &[\bar{s}_3, \bar{s}_9] = \bar{s}_2, && [\bar{s}_4, \bar{s}_8] = \bar{s}_1\bar{s}_3, && [\bar{s}_4, \bar{s}_7] = \bar{s}_2, \\ &[\bar{s}_5, \bar{s}_6] = \bar{s}_1\bar{s}_2, && [\bar{s}_5, \bar{s}_7] = \bar{s}_3, && [\bar{s}_5, \bar{s}_9] = \bar{s}_4, \\ &[\bar{s}_8, \bar{s}_9] = \bar{s}_6\bar{s}_7. \end{aligned} \tag{3.4}$$

(To limit the notation, we have chosen the same symbol "–", although then $\bar{g}$ has two different meanings if $g \in G_1 \cap G_4$. But it will always be clear from the context which one is meant.)

It follows immediately from the preceding relations and Lemma 2.6 that $z_1 = y_1$, $z_4 = x_1$, $Z_1 \cap K_4 = \langle z_1, x_1, \dots, x_5 \rangle$, and $Z_4 \cap K_1 = \langle z_4, y_1, \dots, y_5 \rangle$. As $\{U_2 N_{14}/N_{14}, U_3 N_{14}/N_{14}\}$ and $\{V_2 N_{14}/N_{14}, V_3 N_{14}/N_{14}\}$ are parabolic systems for $G_{14}/N_{14} \cong \Sigma_6$, by conjugation in G_{14} we can identify the generators of G_{14}/N_{14} modulo N_{14}. Taking into account that P_2 and P_3 play opposite roles in G_1 and G_4, we get

$$\begin{aligned} &t_6 N_{14} = s_7 N_{14}, && t_7 N_{14} = s_6 N_{14}, && t_8 N_{14} = s_9 N_{14}, \\ &t_9 N_{14} = s_8 N_{14}, && u_2 N_{14} = v_2 N_{14}, && u_3 N_{14} = v_3 N_{14}. \end{aligned} \tag{3.5}$$

Let $L_1 = \langle t_6, t_7, t_8, t_9, u_2, u_3 \rangle$, $L_4 = \langle s_6, s_7, s_8, s_9, v_2, v_3 \rangle$, $\hat{K}_1 = K_1/\langle z_1 \rangle$, and $\check{K}_4 = K_4/\langle z_4 \rangle$.

From $N_{14} = (M_1 \cap M_4)K_1 K_4$ we get $[N_{14}, Z_4 \cap K_1] \leq K_1' = \langle z_1 \rangle$. So (3.5) implies that the action of L_1 on $(Z_4 \cap K_1)/\langle z_1 \rangle$ is uniquely determined. In particular, $\langle \hat{z}_4, \hat{y}_2, \hat{y}_3, \hat{y}_4, \hat{y}_5 \rangle$ is an indecomposable module for $\overline{L}_1 \cong \Sigma_6$.

TABLE III

c	$\hat{y}_2$	$\hat{y}_3$	$\hat{y}_4$	$\hat{y}_5$	$\hat{y}_7$	$\hat{y}_8$	$\hat{y}_9$	$\hat{y}_10$
t_6	1	1	$\hat{y}_2$	$\hat{y}_3$	1	1	$\hat{y}_7$	$\hat{y}_8$
t_7	1	1	1	$\hat{y}_2\hat{x}_1$	1	1	1	$\hat{y}_7\hat{x}_6$
t_8	1	$\hat{y}_2$	1	$\hat{y}_4$	1	$\hat{y}_7$	1	$\hat{y}_9$
t_9	1	1	$\hat{y}_3\hat{x}_1$	1	1	1	$\hat{y}_8\hat{x}_6$	1
u_2	$\hat{y}_2\hat{y}_3$	$\hat{y}_2\hat{y}_3$	$\hat{y}_4\hat{y}_5$	$\hat{y}_4\hat{y}_5$	$\hat{y}_7\hat{y}_8$	$\hat{y}_7\hat{y}_8$	$\hat{y}_9\hat{y}_10$	$\hat{y}_9\hat{y}_10$
u_3	1	$\hat{y}_3\hat{y}_4$	$\hat{y}_3\hat{y}_4$	1	1	$\hat{y}_8\hat{y}_9$	$\hat{y}_8\hat{y}_9$	1

Now there exists an element $\sigma_1 \in \tilde{G}_1$ such that $o(\bar{\sigma}_1) = 3, [\bar{\sigma}_1, \bar{L}_1] = 1$, $\bar{\sigma}_1^{\bar{t}_1} = \bar{\sigma}_1^{-1}$, and $x_6 = x_1^{\sigma_1}$. If we define $y_{5+i} = y_i^{\sigma_1}$ for $2 \leq i \leq 5$, this definition is independent of the choice $\sigma_1 \in \bar{\sigma}_1$, $\langle \hat{x}_6, \hat{y}_7, \hat{y}_8, \hat{y}_9, \hat{y}_{10} \rangle$ is an $\bar{L}_1$-module isomorphic to $\langle \hat{z}_4, \hat{y}_2, \hat{y}_3, \hat{y}_4, \hat{y}_5 \rangle$, and

$$K_1 = Z_1 \langle y_2, y_3, y_4, y_5, y_7, y_8, y_9, y_{10} \rangle.$$

The remaining relations for the action of $\bar{L}_1$ on $\hat{K}_1$, which follow from (3.5) and Table II, are listed in Table III. Similarly, setting $x_{i+5} = x_i^{\sigma_4}$ for a suitable $\sigma_4 \in \tilde{G}_4$, we get Table IV for the action of $\bar{L}_4 = \langle \bar{s}_6, \bar{s}_7, \bar{s}_8, \bar{s}_9, \bar{v}_2, \bar{v}_3 \rangle$ on $\check{K}_4$.

For further calculations it will be helpful to consider matrices. With respect to the basis $\hat{\mathscr{B}} = \{\hat{x}_1, \ldots, \hat{x}_6, \hat{y}_2, \ldots, \hat{y}_5, \hat{y}_7, \ldots, \hat{y}_{10}\}$ of the $GF(2)$ = vector space $\hat{K}_1$, for the elements $\bar{g} \in \bar{L}_1$ we have matrices of the shape

$$\bar{g} = \begin{pmatrix} g^0 & O & O \\ g^2 & g^1 & O \\ g^3 & O & g^1 \end{pmatrix},$$

TABLE IV

c	$\check{y}_2$	$\check{y}_3$	$\check{y}_4$	$\check{y}_5$	$\check{y}_7$	$\check{y}_8$	$\check{y}_9$	$\hat{y}_10$
s_6	1	1	$\check{x}_2$	$\check{x}_3$	1	1	$\check{x}_7$	$\check{x}_8$
s_7	1	1	1	$\check{x}_2\check{y}_1$	1	1	1	$\check{x}_7\check{y}_6$
s_8	1	$\check{x}_2$	1	$\check{x}_4$	1	$\check{x}_7$	1	$\check{x}_9$
s_9	1	1	$\check{x}_3\check{y}_1$	1	1	1	$\check{x}_8\check{y}_6$	1
v_2	1	$\check{x}_3\check{x}_4$	$\check{x}_3\check{x}_4$	1	1	$\check{x}_8\check{x}_9$	$\check{x}_8\check{x}_9$	1
v_3	$\check{x}_2\check{x}_3$	$\check{x}_2\check{x}_3$	$\check{x}_4\check{x}_5$	$\check{x}_4\check{x}_5$	$\check{x}_7\check{x}_8$	$\check{x}_7\check{x}_8$	$\check{x}_9\check{x}_10$	$\check{x}_9\check{x}_10$

where the g^i are determined by Table I resp. Table IV, in particular,

$$t_6^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \qquad t_7^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

$$t_8^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad t_9^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$w_2^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad w_3^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Moreover,

$$\sigma_1 = \begin{pmatrix} \sigma_1^0 & O & O \\ O & O & I_4 \\ \sigma_1^3 & I_4 & I_4 \end{pmatrix} \qquad \text{with } \sigma_1^0 = \begin{pmatrix} 0 & O & 1 \\ O & I_4 & O \\ 1 & O & 1 \end{pmatrix}.$$

(By I_n we denote the (n, n)-identity matrix and O stands for any matrix (of suitable size) in which all entries are equal to 0.)

Exploiting the relation $[\bar{\sigma}_1, \bar{L}_1] = 1$ successively for the elements $\bar{t}_6$, $\bar{t}_8$, $\bar{u}_3$, and $\bar{u}_2$, matrix multiplication shows that $\sigma_1^3 = O$. (This is equivalent to $[y_{i+5}, \sigma_1] \in y_1\langle z_1\rangle, 2 \le i \le 5$.)

As $\langle t_1\rangle K_1 = Z_4 K_1 = \langle y_6\rangle K_1$, we have $t_1 = y_6 q$ for a suitable element $q \in K_1$. This implies $[Z_4 \cap K_1, t_1] = [Z_4 \cap K_1, q] \le K_1' = \langle z_1\rangle$ and $|Z_4 \cap K_1 : C_{Z_4 \cap K_1}(t_1)| \le 2$. Now L_1 acts on $C_{Z_4 \cap K_1}(t_1)$, $\langle z_1, z_4\rangle \le C_{Z_4 \cap K_1}(t_1)$, and $(Z_4 \cap K_1)/\langle z_1, z_4\rangle$ is an irreducible $\bar{L}_1$-module. Therefore $[Z_4 \cap K_1, t_1] = 1$. Similarly, $|Z_4 \cap K_1 : C_{Z_4 \cap K_1}(t_2)| \le 2$ because $\langle t_1, t_2, t_3, t_4, t_5\rangle K_1 = K_{14} = K_4 K_1$. Since $C_{\bar{L}_1}(\bar{t}_2) = \bar{U}_2 \cap \bar{L}_1$ acts on $C_{Z_4 \cap K_1}(t_2)$ and since we can see from Table III that $Z_4 \cap K_1/\langle z_1, z_4\rangle$ has no $(\bar{U}_2 \cap \bar{L}_1)$-submodule of index 2, we have $[Z_4 \cap K_1, t_2] = 1$, too. The action of $\bar{L}_1$ on $\langle \bar{t}_1, \dots, \bar{t}_5\rangle$ now yields

$$[Z_4 \cap K_1, \langle t_1, \dots, t_5\rangle] = 1. \tag{3.6}$$

As $\bar{\sigma}_1$ is inverted by $\bar{t}_1$, i.e., $\sigma_1^{t_1} \in \sigma_1^{-1} M_1 Z_1$, we get

$$\begin{aligned} [y_{i+5}, t_1] &= [y_i^{\sigma_1}, t_1] = [y_i, t_1\sigma_1]^{\sigma_1} \\ &= [y_{i+5}, \sigma_1] \in y_i\langle z_1\rangle \qquad \text{for } 2 \le i \le 5. \end{aligned} \tag{3.7}$$

Analogous arguments applied to G_4 yield

$$[Z_1 \cap K_4, \langle s_1, \dots, s_5 \rangle] = 1 \tag{3.8}$$

and

$$[x_{i+5}, s_1] = [x_{i+5}, \sigma_4] \in x_i \langle z_4 \rangle \qquad \text{for } 2 \le i \le 5. \tag{3.9}$$

Now we are able to proof

LEMMA 3.1. *Let G be a group of type $\tilde{F}_4$. Then for $i = 1, 4$, there are $\tilde{G}_i \le G_i$ such that $\tilde{G}_i \cong 3^7 Sp_6(2), G_i = K_i \tilde{G}_i$, and $K_i \cap \tilde{G}_i = 1$.*

Proof. By symmetry, it suffices to prove the statement for G_1. We already know that $G_1 = K_1 \tilde{G}_1$ with $K_1 \cap \tilde{G}_1 \le Z_1$. Let $\hat{G}_1 = \tilde{G}_1 / \langle z_1 \rangle M_i \cong 2^6 Sp_6(2)$ and suppose the extension $\hat{Z}_1 \trianglelefteq \hat{G}_1$ does not split. As there exists only one uniquely determined nonsplit extension $2^6 \cdot Sp_6(2)$, we may assume that $t_1, t_2 \in \tilde{G}_1$ are chosen such that $\hat{t}_1^2 = \hat{x}_1$ and $[\hat{t}_1, \hat{t}_2] = 1$ (compare the proof of [Wes, (4.6)]). As above, we have $t_1 = y_6 q_1$ with $q_1 \in K_1$ and $t_2 = \tilde{t}_2 q_2$ for suitable $\tilde{t}_2 \in K_4$ and $q_2 \in K_1$. Now $q_1^{-1} = q_1$ or $q_1 z_1$ implies that

$$t_1^2 = (y_6 q_1)^2 = [y_6, q_1] \in [t_1, q_1] K_1' = [t_1, q_1] \langle z_1 \rangle,$$

and because $\hat{t}_1^2 = \hat{x}_1$ is equivalent to $t_1^2 \in \langle z_1, x_1 \rangle \backslash \langle z_1 \rangle$, we must have $[t_1, q_1] = x_1$ or $z_1 x_1$. Table I together with (3.6) and (3.7) shows that $q_1 = x_6 w$ with some $w \in K_1 \cap K_4$. As $\langle y_6, \tilde{t}_2 \rangle$ is abelian we get

$$\begin{aligned} [t_1, t_2] &= [y_6 q_1, \tilde{t}_2 q_2] = [y_6, q_2][q_1, \tilde{t}_2] && (\mathrm{mod}\ \langle z_1 \rangle) \\ &= [y_6, q_2][x_6 w, \tilde{t}_2] = [y_6, q_2] x_2 [w, \tilde{t}_2] && (\mathrm{mod}\ \langle z_1 \rangle). \end{aligned}$$

But $[w, \bar{t}_2] \in K_4' = \langle x_1 \rangle, [y_6, q_2] \in Z_4 \cap K_1$, and $[t_1, t_2] \in \langle z_1 \rangle$. This yields the contradiction $x_2 = [t_1, t_2][y_6, q_2][w, \tilde{t}_2] \in Z_4 \cap K_1$.

So we may assume that $K_1 \cap \tilde{G}_1 \le \langle z_1 \rangle$. If the extension $\langle z_1 \rangle \trianglelefteq \tilde{G}_1$ does not split, then $\tilde{G}_1 / M_1 \cong 2 \cdot Sp_6(2)$ is the uniquely determined perfect central extension of $Sp_6(2)$ and we may assume that $t_1^2 = t_2^2 = 1$ and $[t_1, t_2] = z_1$ (compare the proof of [Wes, (4.7)] and notice that we can argue in the same way because $\tilde{G}_1$ and $2 \cdot Sp_6(2)]$ have isomorphic Sylow-2-subgroups). As before, we have $t_1 = y_6 q_1$, $t_2 = \tilde{t}_2 q_2$ for suitable $q_1, q_2 \in K_1$, $\tilde{t}_2 \in K_4$. Now $1 = t_1^2 = (y_6 q_1)^2 = [y_6, q_1]$, i.e., $q_1 \in C_{K_1}(y_6) = K_1 \cap K_4$ and $t_1 = y_6 q_1 \in K_4$. Since $z_1 = [t_1, t_2] = [t_1, \tilde{t}_2 q_2] = [t_1, q_2][t_1, \tilde{t}_2]$ and $[t_1, \tilde{t}_2] \in K_4' = \langle z_4 \rangle$, we must have $[t_1, q_2] = z_1$ or $z_1 z_4$. So $q_2 \notin K_1 \cap K_4$ and Table I resp. (3.6) and (3.7) show that $q_2 = x_6 w$ with some $w \in K_1 \cap K_4$.But then $t_2^2 = [\tilde{t}_2, x_6 w] \in [t_2, x_6][t_2, w] K_1' = x_2 \langle z_1 \rangle$ or $x_2 x_1 \langle z_1 \rangle$ and $t_2^2 \ne 1$, again a contradiction and the lemma is shown. ∎

So from now on we may assume that $G_i = K_i\tilde{G}_i$ with $K_i \cap \tilde{G}_i = 1$ and $\tilde{G}_i \cong 3^7 Sp_6(2)$ for $i = 1, 4$. Let $\tilde{G}_1 = \langle t_1, \dots, t_9, u_2, u_3, u_4\rangle$, $\tilde{G}_4 = \langle s_1, \dots, s_9, v_1, v_2, v_3\rangle$, z_i, x_i, y_i, L_1, and L_4 as above. It follows from [Hei, Sect. 3, $(R1)$, $(R2)$, $(R3_0)$, $(R3_1)$] that the relations in (3.2) and (3.4) and Tables I and II hold if we omit the "$-$"'s. Instead of the relations (3.1) and (3.3) we have the following:

$$\begin{gathered} 1 = (u_2u_4)^2 = (u_3u_4)^3 = (u_2u_3)^{12}, \\ 1 = \left[(u_2u_3)^4, \langle t_1, \dots, t_5, t_6t_7, t_7t_8, t_8t_9\rangle\right], \\ \left((u_2u_3)^4\right)^{-1} = \left((u_2u_3)^4\right)^{u_2} = \left((u_2u_3)^4\right)^{u_3} = \left((u_2u_3)^4\right)^{t_9}, \end{gathered} \tag{3.1$'$}$$

and

$$\begin{gathered} 1 = (v_1v_3)^2 = (v_1v_2)^3 = (v_2v_3)^{12}, \\ 1 = \left[(v_2v_3)^4, \langle s_1, \dots, s_5, s_6s_7, s_7s_8, s_8s_9\rangle\right], \\ \left((v_2v_3)^4\right)^{-1} = \left((v_2v_3)^4\right)^{v_2} = \left((v_2v_3)^4\right)^{v_3} = \left((v_2v_3)^4\right)^{s_9}. \end{gathered} \tag{3.3$'$}$$

Next we are going to show that the action of $\tilde{G}_i$ on K_i is uniquely determined.

As above, there is $q_1 \in K_1 \cap K_4 = (Z_1 \cap K_4)(Z_4 \cap K_1)$ such that $t_1 = y_6 q_1$. From $[t_1, L_1] = 1 = [y_5, L_4]$ and (3.5) follows that

$$[q_1, L_1] = [y_6, L_1] \le [y_6, L_4N_{14}] = [y_6, K_1] \le Z_4 \cap K_1.$$

Now Table I implies that $q_1 \in Z_4 \cap K_1$. Thus $t_1 \in Z_4$.

The action of $G_4/N_4 \cong Sp_6(2)$ on the natural module $\check{Z}_4$ shows that all elements in $Z_4/\langle z_4\rangle \setminus (Z_4 \cap K_1)/\langle z_4\rangle$ are conjugate in $O_2(G_{14}/N_4) = N_{14}N_4/N_4$. As $N_{14}N_4/N_4 = K_1N_4/N_4$ and $[N_4, K_4] = 1$, there exists $g \in K_1$ such that $t_1^g \in y_6\langle z_4\rangle$. Now $[Z_4 \cap K_1, g] \le K_1' = \langle z_1\rangle$. So we can replace each $h \in \{y_1, \dots, y_6, s_1, \dots, s_9, v_1, v_2, v_3\}$ by h^g without changing the relations deduced until now. Since the same holds interchanging the roles of G_1 and G_4, there is no loss of generality assuming

$$t_1 \in y_6\langle z_4\rangle \qquad \text{and} \qquad s_1 \in x_6\langle z_1\rangle. \tag{3.10}$$

This implies, in particular, that $[L_1, y_6] = [L_4, x_6] = 1$ and that the actions of L_1 on Z_4 and L_4 on Z_1 are uniquely determined. Conjugating with σ_1 resp. σ_4, we even get a uniquely determined action on K_1 resp. K_4; i.e., we may omit the "$\hat{\ }$"'s in Table III and the "$\check{\ }$"'s in Table IV.

Let $N_0 = (M_1 \cap M_4)(K_1 \cap K_4)$. Then it follows from what we have just said that (3.5) can be improved to

$$\begin{aligned} t_6 N_0 &= s_7 N_0, & t_7 N_0 &= s_6 N_0, & t_8 N_0 &= s_9 N_0, \\ t_9 N_0 &= s_8 N_0, & u_2 N_0 &= v_2 N_0, & u_3 N_0 &= v_3 N_0. \end{aligned} \tag{3.5$'$}$$

Moreover, for $2 \le i \le 5$, we have that $[y_6, t_i] = [t_1, t_i] = 1$. Now (3.6) implies that $\langle t_2, t_3, t_4, t_5 \rangle \le C_{K_{14}}(Z_4 \cap K_1) \cap C_{K_{14}}(y_6) = C_{K_{14}}(Z_4) = K_4$, so

$$\left[t_1, K_4 \right] = \left[\langle t_4, \dots, t_5 \rangle, Z_4 \right] = 1. \tag{3.11}$$

Further, $[t_1, K_1] = [y_6, K_1] = [y_6, K_{14}] \in \langle y_1 z_4, y_2, y_3, y_4, y_5 \rangle$ and (3.7) can be simplified to

$$\left[t_1, y_{i+5} \right] = \left[\sigma_1, y_{i+5} \right] = y_i \qquad \text{for } 2 \le i \le 5. \tag{3.7$'$}$$

Similarly, we get

$$\left[s_1, K_1 \right] = \left[\langle s_1, \dots, s_5 \rangle, Z_1 \right] = 1 \tag{3.12}$$

and

$$\left[s_1, x_{i+5} \right] = \left[\sigma_4, x_{i+5} \right] = x_i \qquad \text{for } 2 \le i \le 5. \tag{3.9$'$}$$

For $7 \le j \le 10$, we deduce from (3.7′), (3.10), and Table II that

$$\left[y_j, y_6 \right] = \left[y_j, t_1 \right] = y_{j-5} = \left[s_{j-5}, y_6 \right],$$

equivalently, $[y_j s_{j-5}, y_6] = 1$. Now

$$\langle x_6, y_7, y_8, y_9, y_{10} \rangle K_4 = K_{14} = \langle s_1, s_2, s_3, s_4, s_5 \rangle K_4,$$

so $y_j s_{j-5} \in C_{K_{14}}(y_6) = K_4$ and $y_j K_4 = s_{j-5} K_4$. From Table II it follows that

$$\left[y_i, y_j \right] = \left[y_i, s_{j-5} \right] = \begin{cases} y_1 = z_1, & \text{if } i + j = 12, \\ 1, & \text{otherwise}, \end{cases} \tag{3.13}$$

which means that the images of elements $y_2, y_3, y_4, y_5, y_7, y_8, y_9, y_{10}$ form a basis of the orthogonal space K_1/Z_1, where $\langle y_1 Z_1, y_{12-i} Z_1 \rangle$ are mutually orthogonal hyperbolic planes. In particular, the action of $\tilde{G}_1/M_1$ on the set $\{y_2 Z_1, \dots, y_5 Z_1, y_7 Z_1, \dots, y_9 Z_1, y_{10} Z_1\}$ is uniquely determined and, in consideration of (3.11), with respect to the basis $\hat{\mathscr{B}}$ we get matrices for

t_1, t_2, t_3, t_4, t_5 of the following shape (again compare [Wes]):

$$t_i = \begin{pmatrix} t_i^0 & O & O \\ O & I_4 & O \\ u_i & t_i^1 & I_4 \end{pmatrix},$$

where

$$t_2^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad t_3^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$t_4^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad t_5^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrices t_i^0 are determined by Table I and $u_1 = 0$ by (3.7′), but we have to calculate the (4, 6)-matrices $u_i = (u_{ijk})$ for $2 \leq i \leq 5$.

First, for $1 \leq j \leq 4$ we have $[t_i, y_{j+6}] \in [K_4, K_1] \leq K_1 \cap K_4$. As $Z_1 \cap K_4 = \langle z_1, x_1, \dots, x_5 \rangle$ this implies $u_{ij6} = 0$. Further, $t_i^2 = 1$ yields $u_i t_i^0 = u_i$ and $u_{2j5} = u_{3j4} = u_{4j3} = u_{5j2} = 0$ for $1 \leq j \leq 4$. Using the fact that $[t_2, U_2] = 1$ and exploiting the relation $[t_2, g] = 1$ successively for $g = t_6, t_8, t_7, t_9, u_2$, we can calculate

$$u_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Now we get u_3, u_4, u_5 from $t_3 = t_2^{u_3}$, $t_4 = t_3^{u_2}$, $t_5 = t_4^{u_3}$. Since $\tilde{G}_1 = \langle t_1, \dots, t_5, \sigma_1, L_1 \rangle$, the action of $\tilde{G}_1$ on $\hat{K}_1$ is uniquely determined. We have listed the corresponding relations in Table V.

The commutators $[u_4, \hat{y}_i]$ can be easily calculated using the relation $u_4 M_1 = t_2^{\sigma_1 t_5} M_1$ and we have omitted the "1"'s in Table V, if we already know that a relation also holds without it. As we have seen above, this is the case for all the generators contained in L_1. By (3.11) and (3.7′) it is also true for t_1, σ_1 and, with respect to y_2, y_3, y_4, y_5, for t_2, t_3, t_4, t_5. So it remains to consider the action of t_2, t_3, t_4, t_5 on y_7, y_8, y_9, y_{10}.

TABLE V

c	$\hat{y}_2$	$\hat{y}_3$	$\hat{y}_4$	$\hat{y}_5$	$\hat{y}_7$	$\hat{y}_8$	$\hat{y}_9$	$\hat{y}_10$
t_1	1	1	1	1	y_2	y_3	y_4	y_5
t_2	1	1	1	1	$\hat{1}$	$\hat{1}$	$\hat{y}_2\hat{x}_3$	$\hat{y}_3\hat{x}_4$
t_3	1	1	1	1	$\hat{1}$	$\hat{y}_2\hat{x}_3$	$\hat{1}$	$\hat{y}_4\hat{x}_5$
t_4	1	1	1	1	$\hat{y}_3\hat{x}_2$	$\hat{1}$	$\hat{y}_5\hat{x}_5$	$\hat{1}$
t_5	1	1	1	1	$\hat{y}_4\hat{x}_3$	$\hat{y}_5\hat{x}_4$	$\hat{1}$	$\hat{1}$
t_6	1	1	y_2	y_3	1	1	y_7	y_8
t_7	1	1	1	y_2x_1	1	1	1	y_7x_6
t_8	1	y_2	1	y_4	1	y_7	1	y_9
t_9	1	1	y_3x_1	1	1	1	y_8x_6	1
u_2	y_2y_3	y_2y_3	y_4y_5	y_4y_5	y_7y_8	y_7y_8	y_9y_10	y_9y_10
u_3	1	y_3y_4	y_3y_4	1	1	y_8y_9	y_8y_9	1
u_4	$\hat{1}$	$\hat{1}$	$\hat{y}_4\hat{y}_7$	$\hat{y}_5\hat{y}_8$	$\hat{y}_4\hat{y}_7$	$\hat{y}_5\hat{y}_8$	$\hat{1}$	$\hat{y}_1$

We have $t_2 = t_7^{u_4u_3}$ and $\|[K_1, t_7]\| = 2^4$. Hence $\|[K_1, t_2]\| = 2^4$ and $z_1 \notin [K_1, t_2]$, which implies $[y_7, t_2] = [y_8, t_2] = 1$. Conjugation in $\tilde{G}_1$ yields

$$[y_7, t_3] = [y_9, t_3] = [y_8, t_4] = [y_{10}, t_4] = [y_9, t_5] = [y_{10}, t_5] = 1. \tag{3.14}$$

Moreover, $[t_2, V_2 \cap L_4] \leq [t_2, U_2N_0] = [t_2, N_0] \leq K_4' = \langle z_4\rangle$. Therefore $[t_2, O^2(V_2 \cap L_4)] = 1$ and Table II and IV show that $t_2 \in \langle z_1, z_4, x_2, x_7\rangle$. (We have $y_6 \in C_{K_4}(O^2(V_2 \cap L_4))$, too, but $t_2 \in y_6\langle z_1, z_4, x_2, x_7\rangle$ implies the contradiction

$$\begin{aligned} z_1x_1x_2 &= [t_1t_2, x_6] = [t_1t_2, s_1] \\ &= [y_6t_2, s_1] \in [\langle z_1, z_4, x_2, x_7\rangle, s_1] = \langle x_2\rangle.) \end{aligned}$$

Now $t_2 \notin K_1$, $x_2x_7 = x_7^{s_1}$, and $[(Z_4 \cap K_1)\langle s_1, \ldots, s_5\rangle, s_1] = 1$. So if we replace each $h \in \{y_1, \ldots, y_6, s_1, \ldots, s_9, v_1, v_2, v_3\}$ by h^{s_1}, then we do not change any of the generators of K_1; i.e., we do not influence the relations for the action of G_1 on K_1. We only have to be careful with y_6 because $y_6^{s_1} = z_1z_4y_6$. This means that, assuming $t_2 \in x_7\langle z_1, z_4\rangle$, instead of (3.10) we can only suppose $t_1 \in y_6\langle z_1, z_4\rangle$. But we will see below that this does not matter. Applying the same procedure with interchanged roles of G_1 and G_4 and conjugating with u_2, u_3 resp. v_2, v_3, we get

$$t_i \in x_{i+5}\langle z_1, z_4\rangle \qquad \text{and} \qquad s_i \in y_{i+5}\langle z_1, z_4\rangle \qquad \text{for } 1 \leq i \leq 5. \tag{3.15}$$

Now (3.15), Table V, and the analogous relations for the action of G_4 on $\check{K}_4$ yield $[t_2, y_9] = [x_7, s_4] \in y_2x_3\langle z_1\rangle \cap y_2x_3\langle z_4\rangle$. So

$$[t_2, y_9] = y_2x_3 \qquad \text{and} \qquad [t_2, y_{10}] = [t_2, y_9]^{u_2} = y_3x_4.$$

Since the corresponding relations for t_3, t_4, and t_5 again follow by conjugation, the action of $\langle t_1, \dots, t_5 \rangle$ on K_1 is uniquely determined. (The relations in Table V indeed hold without the "$\hat{\ }$"'s.) Obviously, the same holds for the action of $\langle s_1, \dots, s_5 \rangle$ on K_4. We resume our result in

LEMMA 3.2. *Let G be a group of type $\tilde{F}_4$. Then $G_1/O_3(G_1)$ and $G_4/O_3(G_4)$ are isomorphic to maximal parabolic subgroups of the simple group of Lie type $F_4(2)$.*

Proof. Let $F \cong F_4(2)$, $F = \langle x_1(1), \dots, x_{24}(1), w_1, w_2, w_5, w_{10} \rangle$ such that the relations in [Gut] hold. It follows from the above results that the maps φ_i: $G_i \to F$, $i = 1, 4$ are homomorphisms onto a maximal parabolic of F with $\ker \varphi_i = M_i$, where the φ_i are defined as in Table VI.

Now we are going to show that we can identify the generators of G_1 resp. G_4 which are contained in $G_1 \cap G_4$. First, from (3.5′) and (3.15) we get

$$t_2 = [t_4, t_7] = [x_9, t_7] \in [x_9, s_6][x_9, N_0] = x_7\langle z_4 \rangle,$$

analogously $t_3 \in x_8\langle z_4 \rangle$, $t_4 \in x_9\langle z_4 \rangle$, and $t_5 = t_9^{v_3}\langle z_4 \rangle = x_{10}\langle z_4 \rangle$. Let $t_i = x_{5+i} z_4^{\epsilon_i}$ with $\epsilon_i \in \{0, 1\}$ and $x = x_2^{\epsilon_5} x_3^{\epsilon_4} x_4^{\epsilon_3} x_5^{\epsilon_2}$. Then $x \in Z_1$ and $t_i^x = x_{5+i}$. It is easy to see that we do not affect the relations for G_1 if we substitute each $h \in G_4$ by h^x. Again we do the same with interchanged roles of G_1 and G_4 and get

$$t_i = x_{i+5} \qquad \text{and} \qquad s_i = y_{i+5} \qquad \text{for } 2 \le i \le 5. \tag{3.15′}$$

TABLE VI

$g \in G_1$	$h \in G_4$	$\varphi_1(g) = \varphi_4(h)$	$g \in G_1$	$h \in G_4$	$\varphi_1(g) = \varphi_4(h)$
x_1	z_4	$x_{24}(1)$	z_1	y_1	$x_{21}(1)$
x_2	x_2	$x_{23}(1)$	t_1	y_6	$x_7(1)$
x_3	x_3	$x_{22}(1)$	t_2	x_7	$x_9(1)$
x_4	x_4	$x_{20}(1)$	t_3	x_8	$x_8(1)$
x_5	x_5	$x_{19}(1)$	t_4	x_9	$x_6(1)$
x_6	x_6	$x_{18}(1)$	t_5	x_{10}	$x_5(1)$
y_2	y_2	$x_{17}(1)$	t_6	s_7	$x_3(1)$
y_3	y_3	$x_{16}(1)$	t_7	s_6	$x_4(1)$
y_4	y_4	$x_{15}(1)$	t_8	s_9	$x_1(1)$
y_5	y_5	$x_{14}(1)$	t_9	s_8	$x_2(1)$
y_7	s_2	$x_{13}(1)$	u_2	v_2	$w_1(1)$
y_8	s_3	$x_{12}(1)$	u_3	v_3	$w_2(1)$
y_9	s_4	$x_{11}(1)$	u_4	–	$w_5(1)$
y_{10}	s_5	$x_{10}(1)$	–	v_1	$w_{10}(1)$

Now $t_1 = (t_1t_2)t_2 = [t_5, t_6]x_7 \in [x_{10}, s_7]x_7[x_{10}, N_0] = y_6\langle z_4\rangle$; similarly, $s_1 \in x_6\langle z_1\rangle$ as before. Consider the map θ_1: $G_1 \to G_1$ defined as follows:

$$\theta_1(x) = x \qquad \text{for } x \in K_1,$$
$$\theta_1(t_i) = t_i \qquad \text{for } i \neq 1, 6, 8,$$
$$\theta_1(t_1) = t_1z_4, \qquad \theta_1(t_6) = t_6x_2, \qquad \theta_1(t_8) = t_8x_3,$$
$$\theta_1(u_2) = u_2z_1, \qquad \theta_1(u_3) = u_3, \qquad \theta_1(u_4) = u_4.$$

By [Gut, (4.6)] the map $\bar{\theta}_1$: $G_1/M_1 \to G_1/M_1$ defined as $\bar{\theta}_1(xM_1) = \theta_1(x)M_1$ extends to an automorphism of G_1/M_1. As $(u_2u_3)^4 = (u_2u_3z_1)^4$, it is not difficult to see that θ_1 extends to an automorphism of G_1. Hence substituting, if necessary, all $g \in G_1$ by $\theta_1(g)$ and all $g \in G_4$ by $\theta_4(g)$ for a suitably defined automorphism θ_4 of G_4, w.l.o.g. we can assume

$$t_1 = y_6 \qquad \text{and} \qquad s_1 = x_6. \tag{3.16}$$

Together with (3.15′) and the actions of L_1 and L_4 on K_{14}, this implies that (3.5′) even holds if we replace N_0 by $(M_1 \cap M_4)\langle z_1, z_4\rangle$. Then $L_1\langle z_1, z_4\rangle = L_4\langle z_1, z_4\rangle$ and $L_1' = L_4'$. Furthermore,

$$s_6 = s_1^{v_1} = x_6^{v_1} = x_5^{u_4v_1} = x_5^{v_1u_4q} = x_8^{u_4q} = t_3^{u_4q} = t_7^q = t_7[t_7, q]$$

for some

$$\begin{aligned} q &= [u_4, v_1] \in O_2(P_1P_4) \\ &= \langle z_1, z_4, x_2, \ldots, x_6, y_2, \ldots, y_5, y_7, y_8, y_9, t_1, \ldots, t_4, t_6, \ldots, t_9\rangle. \end{aligned}$$

Therefore $s_6t_7 = [t_7, q] \in \langle t_2, x_2, x_3, y_2z_4\rangle \cap \langle z_1, z_4\rangle = 1$ and we get

$$s_6 = t_7, \qquad s_7 = t_6, \qquad s_8 = t_9, \qquad s_9 = t_8, \qquad v_2 = u_2, \qquad v_3 = u_3. \tag{3.15″}$$

Proof of the Theorem. Let $F \cong F_4(2)$, $F = \langle x_1(1), \ldots, x_{24}(1), w_1, w_2, w_5, w_{10}\rangle$ such that the relations in [Gut] hold. Let φ_1, φ_4 be the homomorphism from Lemma 3.2 and define a homomorphism φ: $G \to F$ by

$$\varphi(g) = \begin{cases} \varphi_1(g), & \text{if } g \in G_1, \\ \varphi_4(g), & \text{if } g \in G_4. \end{cases}$$

It follows from $G = \langle G_1, G_4\rangle$, (3.15′), (3.5″), and (3.16) that φ is well defined and that φ is an epimorphism with $\operatorname{Ker}\varphi \cap G_i = M_i = G_i \cap M$ where $M = \langle O_3(G_{14})^G\rangle = \ker\varphi$. Moreover, all the relations of $F_4(2)$

which involve only elements that are either all contained in G_1 or all in G_4 are fulfilled in G/M. So it remains to show that $(v_1u_4)^2 = 1$. But, on the one hand, we have $1 = (w_5w_{10})^2 = \varphi(v_1u_4)^2$ and, on the other hand, $P_1P_4/O_2(P_1P_4) \cong \Sigma_3 \times \Sigma_3$. Therefore

$$(v_1u_4)^2 \in \ker\varphi \cap O_2(P_1P_4) \le \ker\varphi \cap S = 1,$$

which is the assertion.

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